An Experimental Study of Mathematics-Learning

By the same author

BUILDING UP MATHEMATICS

THE POWER OF MATHEMATICS

* * *

CONCEPT FORMATION AND PERSONALITY

(Leicester University Press)

An Experimental Study of Mathematics-Learning

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Preface

IT IS a delight to write a preface to this book on mathematical-learning, for Dr Dienes has in these pages illustrated to what extent it is possible by invention to overcome the difficulties associated with the teaching of mathematics. His inventiveness lies in translating mathematical ideas into demonstrable embodiments that are within the grasp of the child. For those of us who have been privileged to see Dr Dienes at work with children it has been astonishing to behold how quickly and surely his mathematical embodiments lead to insights that are at first intuitive and concrete and then more disciplined and rigorous. Only the most hardened traditionalist could resist the beauty and charm of his technique for leading the child into an appreciation of quadratics through manipulating building blocks and balance beams. In the end there is no arbitrariness in the child's sense of the mathematical square and nothing parrot-like in his recital of the concept of a 'set of sets such that the number of elements in each is equal to the number of sets'.

It will not be taken amiss if, in writing an introduction, I comment on the differences between Dr Dienes and some of the others of us who have tried our hand at revising mathematical teaching. Perhaps it can be summed up by saying that Dr Dienes is much more distrustful of 'formalism' than some of the rest of us. The symbols of a language—a natural or a mathematical language—can either be viewed as transparent or as opaque. When we treat symbols as transparent we are principally mindful of the referential function, what they 'stand for' or 'mean'. But it is also possible to treat a symbol system without regard to what lies beyond the symbols in the world of experience, to treat the system as a self-sufficient body of rules for forming and transforming sentences or equations or functions. When a child recognizes that there is a major difference between two languages such that in one $2 \times 4 = 4 \times 2$ but in another *shoe black* = black shoe, then he is treating symbol systems as opaque and exploring the rules whereby the different systems are put together—in this case the rule of commutativity. Many mathematicians today believe that one can indeed introduce many ideas in

mathematics by first giving children a sense of the algebraic or syntactic structure and then moving on to the question of embodiment or reference. Or rather, I should say that many mathematicians hold that there must be a subtle interchange between these approaches. Dr Dienes insists, I think, that the experiential referent of mathematical symbols must always be primary, that the child is always in danger of falling into a non-comprehending formalism. He may be right. What is clear and fortunate is that he is sticking to his guns and in the end that will help us all to determine what is the proper balance between the two approaches.

I have tried to find ancestry for Dr Dienes's effort. One of the ancestral branches is surely the intuitive approach to mathematics that is usually lumped under the heading 'intuitionism'. The other branch is very likely the movements in education usually associated with the names of Froebel and Montessori. The joining of these two great streams of thought has produced a challenging new approach to the teaching of mathematics. Here it is necessary to say a word about the expression 'new approach'. In recent years there has been much misguided debate about the introduction of 'the new mathematics' into the schools. This is a most unfortunate notion, one that has been put forth by the opponents of various modern educational efforts. To the best of my knowledge, none of the major efforts in mathematics-teaching has had this as its objective. For it is an altogether meaningless idea. Mathematics does not divide into 'new' and 'old'. There is only mathematics. The objection of mathematicians to much of the traditional emphasis upon arithmetic and drill is that it is concerned with arbitrary computation or calculation and that it fails to give the learner anything like a proper sense of why things are done as they are. Dr Dienes is concerned with teaching mathematics, the computation coming *en passant*. His achievement is in finding a way to make it clear and intellectually reasonable that order and number and function and identity and the other key matters in mathematics are not arbitrary but, rather, are amenable to the use of reasonable thinking.

I would end with a caution to the reader. Peter Abelard, a towering intelligence in his century, confessed that arithmetic was beyond him. Mathematics, viewed in the perspective of man's cultural history, is a very new invention indeed. We do not know the limit to which ordinary human beings, reasonably taught, can master the chaste rigours of such

powerful artificial languages. If we are not to live in embarrassed dread in the century ahead we will have to understand the mathematical ideas that lie behind the scientific advances that are shaping the modern world—from the DNA code to the binary language of high-speed computers. I see the efforts of Dr Dienes as contributing to the necessary understanding. One can only hope that just as today we expect our children to handle with ease what Peter Abelard found too difficult to master, at some future date we may be equally amused at the mathematical illiteracy that we take for granted as natural today.

JEROME S. BRUNER

Center for Cognitive Studies
Harvard University, 1963

Foreword

THE work described in this book is largely exploratory. Our knowledge of the cognitive processes involved in learning such complex structures as are encountered in mathematics makes it difficult, if not impossible, to mount any large-scale controlled experiments to test a full theory or even a modest number of hypotheses. Our study was undertaken more to help research workers ask relevant questions than to answer these with any certainty. We do offer some tentative eventual answers, but these must be subject to verification by controlled experiment.

The work done by our group came to be called the 'Mathematics-Learning Project'. It consisted of Jerome Bruner and myself, assisted by Helen Kenney, William Hull, Sam Anderson and Eleanor Duckworth. We planned and discussed the work of the experimental groups in frequent seminars held at the Harvard University Center for Cognitive Studies. During these seminars theoretical and practical points were constantly discussed, proposed and rejected. The result of a great number of such meetings (and related memoranda) is reported here.

There were five experimental groups. All members of the Project took part in planning and discussing two of these, referred to in the text as Group A and Group D. Group A consisted of two boys and two girls from a third grade, and each child had an individual observer. Group D consisted of three boys and three girls from a fourth grade. This group was observed as a whole, as the subjects worked either in pairs or in threes. The remaining groups, referred to in the text as Groups B, C and E, were planned entirely by myself and were observed and commented on by William Hull.

During the final discussions on the work of the Project we eventually decided to split the reporting into two parts. A mathematically detailed but psychologically informed account, with educational implications, was to be written by myself. This forms the present five chapters. A psychological evaluation in depth was to be prepared by Jerome Bruner and Helen Kenney, to assess the findings from Groups A and D in the

light of prevalent cognitive theories and to make suggestions for extending these theories in the light of experience gained during the Project.

The piece of work reported in Appendix II describes the stages through which children pass when presented with mathematically structured material with which they are allowed to play freely; this was done at my suggestion by Dr Morino-Abbele, at the Florence Institute of Psychology. Appendix I gives details of the experimental groups and the tools used during the sessions with the subjects. The tables for all the games used in Groups D and E are also given to enable the reader to refer to them as they are discussed in the text.

Z.P.D.

Introduction

FOR thousands of years men have been either discovering mathematics or learning it from their fellows. The discovery of mathematics is essentially different from the discovery of physical objects, since in mathematics what is discovered is some previously unsuspected relationship. Discovery in mathematics is even different from research activity in the natural sciences. In the latter, although we might be guided by theoretical models in terms of which we explain and try to predict future events, the ultimate verification always lies in dovetailing actually occurring physical events with the requirements and consequences of the theoretical model. A physical theory is neither true nor untrue; it either works or does not work. It works if it predicts effectively; it does not work if the predictions are not substantiated. In mathematics the state of affairs is different because the 'truth' of a mathematical statement depends not on any physical correspondence with verifiable events but on whether the statement fits in with the other statements that already constitute the body of known mathematics. We say in mathematics that a statement fits in with some other statements if it is consistent with those statements; that is, if the new statement does not entail a contradiction when considered in conjunction with the other statements. This is a necessary, not sufficient, condition for regarding a new statement as a successful candidate for admission into the body of accepted mathematical statements. For example, the statement '3 is a lucky number' cannot contradict any usual arithmetical statement, as the concept 'lucky' has not been defined to bear any relation to other concepts relating to numbers, except possibly by astrologists. To accept a new statement we require usually that it be capable of proof, by which we mean that there should be a chain of statements each depending in some admitted way on some or all the previous statements; the end of the chain is the new statement. Some mathematical statements may of course be independent of each other without being irrelevant to each other; in such cases new statements may be admitted into mathematics without 'proof', and the consequences of

such admissions or rejections will then be separately investigated. Euclid's fifth postulate is one example: its rejection permits the construction of non-Euclidean geometries; its acceptance entails the usual Euclidean geometry. The continuum hypothesis is another example. Many interesting results follow through assuming the hypothesis, which could not be proved without it. As far as I know the opposite hypothesis has not been investigated to any great extent.

Learning abstract structures must clearly present some particular problems. The fact that mathematics is found difficult or unpleasant or both by so many people testifies to the existence of real learning difficulties. The curious fact is that these difficulties have never been systematically or scientifically studied, and consequently our knowledge of the process of learning mathematics is so scanty as hardly to amount to knowledge at all. The learning theories so far proposed by psychologists provide some simple models in terms of which a certain amount of learning may be explained. But on the whole these models are too simple and elementary to account for the extremely complex processes that must go on in the mind of anyone trying to acquire the complex structures which abound in mathematics.

An excellent start has been made by Jean Piaget and his collaborators in Geneva, though. This has included a thorough-going analysis of the logical foundation of natural number through reconsideration of the ways in which logicians and mathematicians in the past have endeavoured to lay a basis for the notion of natural number. This logical analysis has proceeded side by side with psychological analysis of the process of arriving at a workable notion of natural number, and has been backed up by observation and experiment at all possible stages and ages during which the idea of number is evolved. The experiments are now so well known as to have become classical. It is found that the invariance of the quantity of objects in a group under displacement and rearrangement is not grasped at all firmly until a secure synthesis is made of the cardinal and ordinal aspects of natural number. The ideas of 'one more' and 'the next one', or 'two more' and 'the next but one', take a great deal of time to become synthesized into a workable structure, and it is not until this has happened that recursion can be used in a way that is truly operational. There are many stages along the road before this synthesis is achieved: very important and interesting work is being done in Geneva on the investigation of these stages by

Gréco; a successful attempt to formalize the structure of natural number corresponding to the ways in which children in fact naturally proceed was recently made by Grize. Grize has given a formalized version of the idea of '*groupement*', a term coined for the curious kind of rather difficult structure the natural numbers appear to be, in contradistinction to the idea of 'group', a mathematically much simpler but psychologically much more mature idea.

These lines of research have so far concentrated on the elementary, but of course extremely fundamental, processes of forming the early mathematical concepts in children. Another approach would be to analyse mathematical thinking into its components, logical as well as mathematical, and then investigate experimentally how subjects do in fact acquire the corresponding logical (or mathematical) concepts. The study by Bruner, Goodnow and Austin[1] was a detailed attempt to examine the strategy of subjects in forming certain simple logical combinations, such as conjunctions and disjunctions. However much light such studies may shed on detail, it is clear that mathematical or any other kind of reasoning is more than the sum total of its parts. In order to gain a complete view of mathematical thinking it is first necessary to watch such thinking in action, and it was realizing this which gave rise to planning the present study. Since no systematic observations of mathematical learning *in situ* existed, as far as we knew, we had first to record such observations. We hoped, from detailed study and discussion of these, to formulate principles which could then be incorporated in a theoretical model. Such a model would act as a preliminary one on which a full learning theory applicable to mathematics-learning could be based.

There was very little to go on in planning experimental sessions. It was eventually decided to lean heavily on the pragmatic experience gained with several thousand children in the Leicestershire Mathematics Project, where they had learned through their individual experiences. This had the advantage of making it possible to predict approximately what could be learned.

The observer assigned to each subject could record his subject's attempts with the physical material—an added experimental advantage as it would thus be easier to extrapolate to these subjects' underlying thought processes than if one were simply observing a subject silently

[1] *A Study of Thinking*, Wiley (New York, 1956).

trying to solve a problem. It was in this way that the Harvard Mathematics-Learning Project was born. The Project was to be run jointly by Jerome Bruner and myself, combining Bruner's expert knowledge and experience of thought processes in general with my own detailed knowledge of the structures we were to experiment with and my experience of children learning such structures from physical embodiments. The work of the Geneva school was of course to be implicitly kept in mind throughout, as well as the contributions of Sir Frederick Bartlett[1] and others in the field. It was happily possible to keep in touch with the Madison Project, and Robert Davis's ideas on symbol games were also incorporated eventually into the work by turning groups and vector spaces into games and playing these games with two of the experimental groups.

We assumed throughout that a certain freedom to 'fiddle around' was essential to successful formation of mathematical concepts. This freedom was given in varying degrees to the different experimental groups. Its use is what we describe as 'play' in the ensuing pages, and this is why the first chapter is concerned with play. The next three chapters deal with the functioning of abstraction, generalization, symbolization and allied processes, seen through examples quoted from the protocols of the experimental groups. The educational implications of the work are discussed in the fifth chapter, and at the end there is a brief survey of the present position of enquiries into the problems related to mathematics-learning.

[1] *Thinking*, Allen & Unwin (London, 1958).

I

On the function of play in mathematical thinking

1. *Types of play*

Much has been written about play and its function in a child's life. We are concerned with the function of play only as it affects, positively or negatively, the cognitive processes in which children engage. It is almost impossible to define play; and in any case definition assumes something more fundamental than the object of the definition, in terms of which the definition is framed, and it is worth pondering whether there are many or indeed any more fundamental phenomena than play which one might want to use in describing the behaviour of children. It is usually easy to agree whether a child is playing or not in any given situation, and although there are sure to be boundary cases where the decision is uncertain, nevertheless I do not propose to attempt here an explicit definition of play. This does not mean that the circumstances which are categorized as play will not be described or delimited; this will be necessary for the sake of clarity. One should remember, however, that explanations are not the same as definitions.

Many people have suggested before now that the natural way in which children acquire knowledge is through play in some form or another. It is not at all clear, however, what the processes are that lead from play to the purer forms of cognition such as construction of classifications, generalizations, logical classifications and deductions. In our project we observed over shorter or longer periods of time a number of small groups of children engaged in learning mathematical concepts from experiences with materials. Some of the protocols of our observations are going to be used here to formulate hypotheses about the role of play in cognitive activity, as well as other hypotheses about the interrelation of the various forms of cognitive activity usually engaged in by human beings. It will be useful to begin with a preliminary

classification of different kinds of play, with examples from the protocols, and afterwards to discuss their function in the whole cognitive process.

It is tempting to distinguish between primary and secondary play. Primary play is playful activity directly resulting from the gratification of certain drives or instincts; in this case the pay-off is immediate, namely gratification. Secondary play could be defined as playful activity in which a certain amount of time-perspective is noticeable in the execution of plans, and the subject shows an increased awareness of his own activity. Pay-off here is delayed, and is not necessarily the gratification of any drive or instinct, but the achievement of a definite aim of which the subject is aware. The types of aim would no doubt be determined by gratification of fundamental drives in the first place, which would be the original agents that had helped to build in the behaviour patterns exhibited by the subject. According to this view, secondary play is further removed from the primary source of energy in the subject than primary play. The difficulty is to say when play is still close enough to the source to be termed primary, and when it is sufficiently well-organized a behaviour pattern to be called secondary. There is another difficulty—the affective loading usually associated with the word 'primary 'by dynamic psychologists. Curiosity, shown in exploratory behaviour in animals as well as in humans, is just as likely to be fundamental and close to the original supply of energy as other drives such as sex or hunger, and may not be loaded affectively at all. For these and other reasons it is probably better to speak in terms of a continuously or at any rate gradually changing degree of awareness, rather than of a dichotomy between primary and secondary processes.

Let us now survey briefly the kinds of play which may be relevant to mathematics-learning, both as helpful and as interfering agents on the road towards understanding mathematical structures.

There are at least three rather different forms of play. One is the kind already mentioned, the exploratory-manipulative type of play. Children like the feel of objects, they like to run their fingers down surfaces, to put things together and take them apart. Starting with this curiosity-prompted play, they may proceed to become aware of the properties of the material they are playing with, ask questions about what can or cannot be done, and so generally move up along an

awareness or *time-perspective* axis. But it is unlikely that play will remain purely manipulative for long. There are at least two obvious directions in which a child might proceed, having had his fill of purely manipulative-exploratory behaviour. One is representation; the other is the search for regularities.

Representational play will arise as soon as the objects children are playing with begin to stand for something they are not. Another component is added to the activity, namely imagination. Manipulation with imagination added will result in representational play; the objects may stand for animals, people, houses or anything else at all. There may be some attempt to put the objects together so that the result resembles the thing imagined, or there may be pure representation where the objects stand for things by definition or assignment. The physical properties of the objects will no doubt influence the kinds of representations that will be used, and often, conversely, problems arising out of making the representations more realistic will result in a resort to further manipulations of non-representational kind.

Manipulative play may quite imperceptibly move over to a search for regularities. When a rule or rules are found, play may occur which uses these rules. Children delight in regularities, and the formulation of a rule-structure is a kind of closure which ties up all the loose ends of past experience. Children feel safe within such a rule-structure. Once it has been thoroughly mastered, and has become part of the currency of play, the closure is often reopened by asking questions. In this case the rules themselves will be the objects of manipulation and the situation is open again at a higher level.

There will of course be mixed variants of these kinds of play. It should not be thought that I imagine hard and fast boundaries between such categories. On the other hand we must not confuse the act of playing a game which has two or all three ingredients with the act of passing from one type of play to another. The changing back and forth between different types of play may step up the energy for the more advanced cognitive processes which arise out of play, much as, for example, alternating current makes the exploitation of electrical energy easier under certain circumstances.

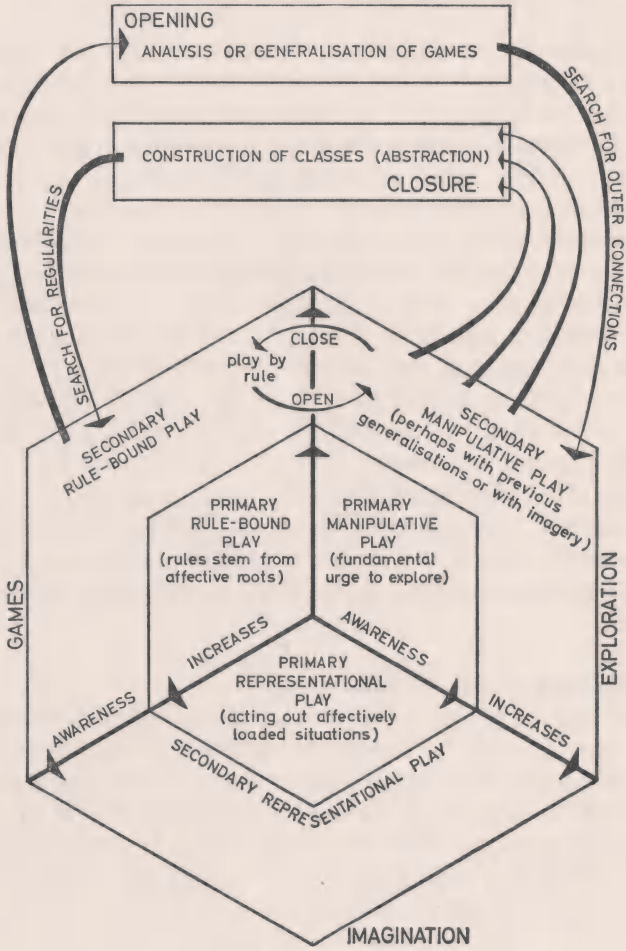
To sum up, we shall speak of manipulative play, representational play and rule-bound play. *Manipulative play* is an activity which could be described as exploration, since the subject is hardly aware of the

exploratory process in the beginning, but awareness increases with the accumulation of experience. *Representational play* takes place when objects or people are assigned properties different from those they in fact have. Imagination is an essential component of this kind of play. *Rule-bound play* is essentially 'playing a game', meaning that choices are limited in some circumscribed way by the rules of the game.

Let us examine in a little more detail the change-over from manipulative to rule-bound play and vice versa. Manipulation is possible with an endless variety of objects available to us in our environment. These objects differ from one another in shape, size, colour and a host of other attributes. Yet certain types of manipulation are always possible under enormously varied conditions. For example, we can have any two sets of three objects and always arrange them in corresponding pairs so that every pair has an object from each set. The properties of the objects are just not relevant for this manipulation; all the situation needs is that the objects be discrete and manipulable. Out of this variety of manipulations we construct the concept of three, and we give it the name three. The natural numbers arise in the first place as a kind of common currency by whose means we can express the relative sizes of collections of objects.[1] Each natural number is a closure round all the experiences that have led up to realizing there must have been something the same about these experiences. A closure is not a composite of all the experiences, as such a composite would represent a cognitively intolerable load. The reason behind constructing concepts such as three is, on the contrary, the need for cognitive economy. The closure represents not a composite, but a distillation of what is essentially the common link between the relevant experiences. We construct the class 'three', whose elements are collections of three objects, by a distillation process usually known as abstraction. An abstraction serves to organize our future experiences so that we can identify future collections of three objects as elements of this class whose properties (or at least some of them) we know. This knowledge enables us to make accurate predictions about what can be done with such collections. The construction of classes is the result of the discovery of regularities in a wide

[1] This is only one aspect of natural number referred to by Piaget and his co-workers as *cardination*. Piaget maintains that unless ordination is also grafted on to this concept, whole numbers cannot be reasoned about in truly mathematical contexts.

variety of situations, and the defining attributes that have been associated with the classes formed will be the newly-discovered rules which sort the chaos of seemingly disconnected experiences into something like order. Thus we have moved over from manipulative play to rule-bound play, but on the way we have engaged in a cognitive activity which could hardly be described as play: we have constructed classes from their elements (abstraction). The diagram below shows a number



of arrows leading out of the area marked manipulative play into the closure, or construction, of classes. Once a closure has been achieved rule-bound play can take place, making use of the rules just established. I shall distinguish between a rule which is abstract and a rule which is seen to operate only in the narrow field in which it has been discovered.

Once classes have been constructed and the regularities establishing membership of the classes have been mastered, questions might be asked about the rules, or other rules proposed. This is an attempt at reopening the closure established by the class-formation. Analysis of the rule-structure or its extension might well follow. This is indicated on the diagram by the arrow issuing from the area marked rule-bound play to the opening, which is the analysis, extension or other *use* of the rule-structure. This again can hardly be described as playing, being a fairly complex cognitive activity. It does, however, make play on a different level possible. The analysis of the rule-structure often leads to further manipulative play with the established rules or with other rules which could in turn lead to the formation of further classes. The cognitive work, according to this model, consists of bridging the two kinds of play, and each time the bridge is crossed the play that takes place is on a higher level (more will be said about these levels later). I suggest that energy is released for cognitive work every time play takes place, and so the power supply never dries up and the formation of classes of a higher and higher order goes on apace. In some cases representational play may come into these cycles, but it does not seem long before the rules in the 'stories' become more important than the stories themselves. Examples of this kind of shift will be given from the protocols.

2. *The function of different types of play*

If play and the higher cognitive activities are indeed intimately connected, then it should be possible to make use of this fact in the promotion of learning by manipulating the play variable. This is an educational problem, and before it can be satisfactorily tackled the psychological problem of the relationship between play and learning will need to be clarified. The most obvious way to start experimentation is to create situations in which children can be observed from the point of view of a naturalist. We did this by providing children with some

concrete material, structured to correspond to certain mathematical concepts, in a permissive atmosphere in which they could feel free to explore. We used material previously tried out in England (the Multibase Arithmetic Blocks), and some predictions about the probable behaviour of the children were possible. Written instructions, supplemented by verbal ones where necessary, were given to the children at times on what to do with the material. There were five groups of children, Groups A, B, C, D and E. (For details see Appendix I, *Description of the experimental groups*.) With the exception of Group B, children were not told explicitly either to play or not to play with the material; in Group B they were told they could do anything they liked.

2.1. *The use of free play—manipulative or representational*

The role of play can be varied by the way the teacher introduces the material. I have suggested that there are natural ways of passing from manipulative or representational play to the rule-bound type of play. Such transition can be artificially induced by suggesting problems to children after they have had a certain amount of free manipulative or representational play with any given material. If the problems are suggested and not insisted upon, a great deal of the natural desire to explore can be preserved, and at the same time a certain amount of direction introduced. This was done with Group B. Most of the play appeared at first to be representational, though some was purely manipulatory; some, on the other hand, was more projective in character. Play which began with a 'story framework' soon came to be motivated by the structure of the problems rather than by the story content, though some children continued to regard the representation as of major importance.

In this work with Group B we decided to start with the least structured procedure possible, i.e. with free play. Only the rectangular version of the Multibase Arithmetic Blocks[1] material was introduced at first, the base varying from 3 to 6 inclusive, and powers going up to 6 (see Appendix I). Play was entirely free to begin with and the children were extremely busy constructing castles and other structures of various kinds. But it was not long before they began to calculate how many unit cubes there were in the various pieces, and some tried to find the totals for whole boxes of the material. Only two or three children in the

[1] This will be abbreviated in future references as MAB.

class, however, engaged in this activity (the whole class consisted of 21 children). There was one child in the class, let us call him George, who seemed to need to use the blocks projectively for working out emotional problems. This resulted in some aggressive games of a warlike character which threatened to disrupt the class on more than one occasion. While there was any free play going on anywhere in the classroom it was impossible to deflect George from these projective activities.

There were a number of children, approximately two-thirds of the class, who responded positively to the idea of doing some problems. Children were asked in turn if they would like a problem to do, and, if the answer was in the affirmative, problems of various kinds were provided. To describe these problems, the blocks will be denoted by the following abbreviations:

$1 \times 1 \times 1 = U$ (Unit)
$1 \times 1 \times N = L$ (Long)
$1 \times N \times N = F$ (Flat)
$N \times N \times N = B$ (Block)
$N \times N \times N^2 = LB$ (Long Block)
$N \times N^2 \times N^2 = FB$ (Flat Block)
$N^2 \times N^2 \times N^2 = BB$ (Block Block)

A suffix, when appended, will indicate the value of the base used. For example FB_3 would mean $3 \times 3^2 \times 3^2$. Here $N = 3$.

(1) *The simple prison problem.* The shape of a B must be built with pieces out of the same box, but with a hollow inside. There must be a floor, a roof and 4 walls, but no doors or windows.

This problem caused considerable difficulty, as it forced the children to be analytical about the relationships present. They were so unable or unwilling to do this at first that many of them declared the problem was impossible. When the problem was solved it was further refined by requiring the children to use as few pieces as possible.

Then the U's (units) were identified with the prisoners, and the question put how many prisoners it was possible to pile into the prison. This produced some useful learning for $N = 4$, 5 and 6.

(2) *The compound prison problem.* The shape of 2 B's must be built with the pieces out of the same box, but with a hollow inside. There must be a floor, a roof and 4 walls, but no doors or windows.

This was increased to 3 B's, then to 4 B's.

The children who had done the simple prison did not seem to find the compound prison difficult, although sometimes they would just make two prisons and put them beside each other, thereby making a superfluous double wall. No child put the 4 blocks in the form of a square at first, but continued to make the prison a long one. The square shape was eventually suggested to them, and they were very surprised to find that more prisoners could be fitted into the same 4 blocks when these were placed in the form of a square than if they were put end on. One wonders whether, if children encountered these situations sooner than they do now, and in a more concrete way, there would be less confusion between area and perimeter, and surface and volume.

A good half of the class were very intrigued with this and other problems, and would eagerly ask for more, and harder, problems. The following were also given:

(3) *The simple tunnel problem.* The shape of a B must be built with the pieces out of the same box, but there must be a tunnel right through the middle of the block, just big enough to hold one L if N were odd, and just enough to hold 4 L's if N were even.

Here again, many children pronounced the problem impossible after some trials, but when assured that it was indeed possible they persevered and usually reached the solution.

(4) *The double tunnel problem.* Same as previous problem, except that two horizontal tunnels one on top of the other were now required right through the centre.

This was a little more difficult, but there was great rejoicing when the problem was solved.

This problem-solving went on side by side with free play, no pressure being exerted in either activity to direct the children, except that George had to be restrained repeatedly in the interests of the rest of the class. This lasted about three weeks, and was followed by the Christmas vacation. We felt that the children were quite familiar with the material, that they had had enough free play, and were ready to use the implicit knowledge gained to make the relationships embodied in the material more explicit. We also thought they were probably capable of expressing some of these relationships in symbolic form.

During the last session before the vacation the higher powers of the

bases 2, 3, 4 and 5 were introduced up to the twelfth power in base 2, ninth in base 3, and the sixth in the bases 4 and 5. These very large pieces, however, had the effect of stimulating the children's imaginations in the direction of representational games. They were not used by any children for generalizing what had been learned with the smaller powers. For this reason we decided to restrict the powers to the sixth in bases 2 and 3, and to the fifth in bases 4 and 5 for the ensuing sessions.

This is an example of a group of children starting with the use of structured material in an entirely free way and being led, either by the nature of the material or by suggestions from the 'teacher', into enquiries about some mathematical structures easily represented by such material. It must not be thought that these children were as yet learning mathematics proper; they were, however, having experiences which would facilitate further learning later. The play element was providing the motive power for this pre-mathematical activity.

It appears from this example that to achieve effective mathematical learning, at least in certain instances, it may be necessary to exploit one type of situation fully before introducing another type and thereby confusing the developing outlines of the first situation. The children in the example were quite happy investigating the structure of the material they had, until the higher powers were thrown in. This made the situation again quite irregular. It was not at all clear to the children that the new pieces were in fact extensions of the old. A kind of regression took place, from which they had to build up again to the point where they would be interested in new relational possibilities opening up, rather than the representational ones for which there had suddenly appeared a greatly increased scope. The situation would no doubt have been different had the larger pieces been available to the children all the time.

Regression to a more primary type of play may ensue for emotional reasons. Examples of such regressions are given at the end of this chapter. The account just given here is one where the regression took place not for emotional reasons but because of a change in the external circumstances of the situation in which the children found themselves. The exploratory drive, as well as the children's imagination, was stimulated by the change, and presumably this was responsible for the regression from the higher-order process of problem-solving to the play that ensued. It is very likely that the course of events would have

been different had we decided to wait for the children to make the first move towards structuring the situation. It has in fact been found in some cases when children are left alone with the material that some of them eventually begin to ask questions relating to structure such as 'What happens when you put three blocks together, couldn't we have a piece for that?' Such feelers could then be answered by producing the requisite piece, which would not have been made available until the enquiry was formed.

There is a teaching problem here of the utmost importance, namely how to decide whether or not to interfere with the natural process of concept-formation, and how best to interfere if we have to. The attendant psychological problems—what happens to the mental processes as a result of interference—are no less intractable: a great many studies on the relationship between learning and teaching are needed before the position can possibly become clear. As a result of the tentative experiments reported here it is not possible to do more than pinpoint some problems and suggest a theoretical model in terms of which further hypotheses may be framed and then tested.

2.2. *The use of rule-bound play*

Of the three types of play discussed, clearly the rule-bound type is the most suitable as a point of interference or 'teaching' in order to use the play energy in the service of higher-order cognitive activity. Both manipulative and representational play may be considered as pre-rulebound, since rules cannot be extracted from the environment without first interfering with (i.e. testing) the environment. There is a great difference between children *discovering* the regularities through manipulation, and being *given* a game which already has the rule-structure that they themselves were eventually intended to learn and abstract.

Groups D and E on the whole had the prepared games treatment. Group A were expected to find the regularities from the special experiences to which they were subjected. Group B had the longest period of 'free' play, followed later by specially designed abstraction exercises. Group C had a mixed diet of concept-construction experiences and prepared games. A description will follow later of how mathematical thinking may be viewed as a living thread woven out of the main ingredients of concept-construction, games and analysis; this might eventually be helpful in the problem of sorting out the function of play

in the process of learning mathematics and perhaps of learning any abstractions. We certainly found that the mathematical game had a great deal of motivating power, perhaps even more than the concept-construction experiences. If it is the case that the energy released by play can be tapped through rule-bound play, there are limitless opportunities and possibilities in planning the mathematical life of children. We shall give an example of children in Group E going through three successive cycles of learning, but before we do so a few words must be said about such 'cycles'.

Any statement, mathematical or otherwise, says something about something. There is always a subject and a predicate. One has thoroughly mastered a predicate if one can recognize any relevant situation in which the predicate is applicable and, where it is applicable, can decide whether what the predicate states about the particular subject is true or not. Putting it philosophically, the universe of discourse of the class of things to which the predicate is applicable must be a well-defined class for us, and whenever we apply the predicate to an element of this universe we must be able to say whether we are making a true statement or not. When we have reached such a stage in relation to a predicate we can say that we have completed a cycle. This being so, the predicate will have become common currency in our conversation or work, and in talking about it this predicate will become a subject of further statements, so that what we are stating about it will become the new predicate. Thus we have begun the second cycle, which we may eventually complete by satisfying these same requirements. Mathematical thinking differs in this way, among others, from ordinary thinking, in that the thinker may be required to complete such cycles in rather rapid succession. In everyday life we take our time in forming new predicates, in the hope of sorting out our physical environment more efficiently than before.

The cycles of mathematical thinking performed by some of the children in Group E are given below:

Example 1 Group E Vector spaces

First cycle. Consider any number of objects of various kinds. Take, for example, cups, boxes and gloves. Each object will have two positions,

which are going to be considered 'opposite'. Cups can be right way up and upside-down, boxes can be open or closed, gloves can be right side out and inside out. If we put any number of these objects on a table we shall say that we have a vector (strictly speaking, the representation of a vector), and we shall describe the vector simply by saying how many objects of each kind we have on the table. There is one proviso, namely that we do not count pairs of opposite objects. For example:

One open box, two inside-out gloves, three right-way-up cups will be considered the same vector as (3 open boxes, 2 closed boxes) and (3 inside-out gloves, 1 right-side-out glove), (6 right-way-up cups and 3 upside-down cups).

As the opposite pairs in the second arrangement are discounted, we have the previous arrangement by convention.

The elements of the 'universe of discourse' are now vectors, which are collections of a certain variety of objects, each variety occurring in one of two 'opposite' positions, and pairs of opposites 'cancelling out'. Addition and subtraction of vectors can be defined as the joining of further objects and the removal of objects. Subtraction can always be carried out since if there are not enough objects to take away, we can always imagine enough opposite pairs. The world thus created is a vector space, in which the number of dimensions is equal to the number of the varieties of objects we choose to consider. In the above example, therefore, there are three dimensions.

Second cycle. We can now regard a whole vector space as our subject. We may then start doing something to our vector space; transforming it, for example, into another vector space. The predicate in the second cycle will be the transformation, the subject being the vector space.

Take a two-dimensional vector space. If we take pieces of paper with red arrows and blue arrows on them we can have a red dimension and a blue dimension. The arrows should always point either towards us or away from us. Let us write

R_a for red arrow away,	R_t for red arrow towards,
B_a for blue arrow away,	B_t for blue arrow towards.

The away and the towards arrows of the same colour will cancel out in the same vector, so that for example

$$2B_a+3R_t+5B_t+6R_a \text{ is really only } 3B_t+3R_a$$

Let us now have two tables; we can call them tables X and Y. On table X we can put down any vector, for example ($3B_a$, $2R_t$).

We now assume the following 'dictionary' for 'measuring' our vectors:

The 'away' cards are counted with squares.

The 'towards' cards are counted with triangles.

Triangles and squares cancel out after 'measurement'.

To find out what to put on table Y we must look and see what is on table X.

$$3\times\begin{bmatrix}\text{no. of red ones}\\ \text{on table X}\end{bmatrix}+2\times\begin{bmatrix}\text{no. of blue ones}\\ \text{on table X}\end{bmatrix}=\begin{matrix}\text{no. of red}\\ \text{ones on Y}\end{matrix}$$

$$\begin{bmatrix}\text{no. of red ones}\\ \text{on table X}\end{bmatrix}+\begin{bmatrix}\text{no. of blue ones}\\ \text{on table X}\end{bmatrix}=\begin{matrix}\text{no. of blue}\\ \text{ones on Y}\end{matrix}$$

If something comes out as squares we place the arrows 'away'; if it comes out as triangles we place the arrows 'towards'.

For our chosen vector ($2R_t$, $3B_a$) this would give us

$$3\times(2\text{ triangles})+2\times(3\text{ squares})=6\text{ tr.}+6\text{ sq.}=0\text{ (i.e. no reds)}$$
$$(2\text{ triangles})+\quad(3\text{ squares})=1\text{ square (i.e. 1 blue away)}$$

so if table X has 2 red arrows towards and 3 blue arrows away, table Y has 1 blue away.

We could work out some others and see how the things on table X get turned into things on table Y. We could also choose someone who does not know what is on table X and just tell him (or her) what had to be put on table Y. How would he (or she) work out what must have been on table X?

Third cycle. Having become familiar with transformations we can start doing something to them; for example we can try to invert them. The predicate 'transform' will now become the subject 'transformation', on which we shall operate with the predicate 'invert'.

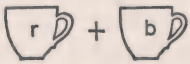
We have been using the transformation:

$$3\times(\text{no. of reds on X})+2\times(\text{no. of blues on X})=\text{no. of reds on Y}$$
$$(\text{no. of reds on X})+\quad(\text{no. of blues on X})=\text{no. of blues on Y}$$

We might find it easier to use some measuring things. Let us measure the reds with red cups and the blues with blue cups. Put as many things in the red cups as we have reds in our vector, and as many things in the

blue cups as we have blues in our vector. Away arrows are squares again and towards arrows are triangles, as before.

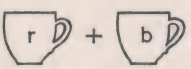
=no. of reds on Y



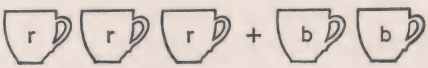
=no. of blues on Y

Take a vector, say 4 red away arrows and 5 blue towards arrows. Put this on X. The away arrows are squares, so we put 4 squares in each of the red cups. The towards arrows are triangles, so we put 5 triangles in each of the blue cups. The first row will give us 12 squares and 10 triangles, which is 2 squares. So 2 red away arrows will go on table Y. The second row will give us 4 squares and 5 triangles, which is 1 triangle, so 1 blue towards arrow will go on table Y.

How could we have done this backwards?

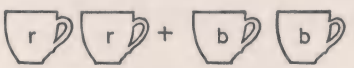
We know that 

=1 blue towards on Y



=2 red away on Y

From the first one of these we might well think that



=2 blue towards on Y

The second arrangement has one red cup's worth of squares more than the third arrangement, so

(2 blue towards)+(1 red cup's worth)=2 red away

but 2 towards+4 *away*=2 away, so there must have been 4 squares in the red cup. We could now find a way of working back to the blue cups.

The first cycle has 'vectors' for its subjects, and such things as adding and subtracting vectors for its predicates. Strictly speaking this is preceded by a short preliminary cycle in which the creation of vectors out of objects takes place. In this the objects are the subjects of our statements, and how many of each we put on the table are the predicates.

The second cycle has the whole vector space for subject, and the

transformations are the predicates. Instead of doing something to a vector (like adding another vector), we do something to *all* the vectors, i.e. to the whole space. By such 'doing' we obtain another space. This process is a transformation.

The third cycle starts by 'doing' something to transformations, i.e. in the example quoted the transformations are inverted. The end of this cycle was not reached by the children in Group E, as this would have meant an awareness of how to invert *all* transformations (if possible). A further cycle could be started by studying the properties of inverse transformations.

These extremely complex structures were all studied as games by children in Group E and it was the very complexity of the structures that provided motivation for the work. Any suggestion of making the games harder was always greeted with enthusiastic agreement. The level of motivation was uniformly high and the excitement of the games did clearly arise from their structure, as there was no 'cover story'. There seemed to be, however, a natural reluctance to proceed to the next cycle before the previous cycle had been completed. Remarks such as 'Let's do some more from X to Y before we do it the other way' testified to this. Members of the group eventually went on to 3 by 3 transformations and their inverses; it appears that the reason they were successful with such complex mathematics was partly that the game-quality of the activity tapped more energy, and partly that they were learning from their own experience.

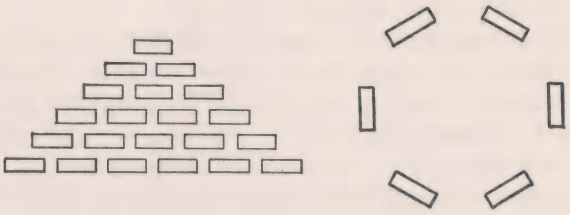
2.3. *The use of representational play*

We might ask whether representational play may be brought in to serve the ends of mathematics-learning. Such play can be used in two obvious ways. One is by inventing a story which is structured mathematically to provide the stimulus to investigate this structure. Another is to take advantage of the hidden mathematical qualities of children's representational games and guide the children by asking them questions *about their game* whose answers would lead them to discover mathematical structure.

An example of representational play paying off quite handsomely from the mathematical point of view is the following, taken not from any of the five experimental groups, but from the remainder of the third-grade class from which Group A were taken.

Example 2 Classroom group The tower game

Some of the children in the class were doing structured exercises from written instruction cards and some were playing freely with the material. Some of the latter were building 'walls' coming to an apex with squares, whose vertical cross section would look something like the figure below, left:



We suggested to these children that they should curve the wall round so that it looked like the figure on the right when seen from above, but with the other layers on top. Then that they built another identical curved wall. They were now asked if they could build, with exactly what they had in both their walls, a round tower which was the same height all round, closed, and exactly as high as the highest part of their original wall. Some children did not respond, finding the instructions too difficult, but others responded and eventually reached a stage where they saw that there would always have to be one more square per storey than the number of storeys. These children had clearly taken a step in the direction of summing arithmetical progressions.

In the last example the play subsequently directed along mathematical lines was relatively free play—free also in the sense that the material with which these children were playing did not even unconsciously compel them in any particularly 'suitable' directions in line with the design of the material. The objects played with were squares, and so the number of possible avenues for play was practically unlimited. Some material would have a more limiting influence on play, as its very construction would probably suggest certain ways of using it. The following

example is one showing how the play material, coupled with a suitable 'cover story', can have a powerful influence on the course of a subject's thinking.

Example 3 Group C The family game

The idea of the family, mothers, fathers, children and babies, was used as a 'cover story' to get the children started. The game was intended to be an exercise in the use of powers.

The children were given the logs and indices series to the base 2, and told to make as many people as they could with the pieces; some would be small and some large, but they must all have the same shape. Finally the people were constructed out of four pieces each; the head was a 2^{3n} piece, the legs were two 2^{3n+1} pieces, and the body was one 2^{3n+2} piece. With the timber at the children's disposal this was possible for $n=0, 1, 2, 3$. They became the baby, the child, the mother and the father.

The game consisted in finding out how many babies you needed to make a whole child, how many children to make a mother, how many babies to make a mother and so on. Then the game went on to find out how many of one kind of person were needed to cover up the front of another kind of person entirely. These yielded the even powers, the 'making' yielding the powers divisible by 3 (i.e. where the exponents were divisible by 3).

On one occasion one of the children, when asked to build the families (the other bases were of course also used to build other families), placed mother neatly in father's lap, the child in mother's lap and the baby in the child's lap. This apparently primary representational game, which arose spontaneously as one way of playing the game, did not have a regressive effect on the cognitive processes taking place. The same subject discovered for example that you could use 2^5 babies with which you could make enough children to cover the whole front of the mother.

Instead of structuring the material to induce certain lines of mathematical thinking it is possible to structure games mathematically with the same end in view. The games can be played with real objects or in the children's imagination. In the former case we should be making use of

representational play structured to serve mathematical ends. Two examples of such representational play are given below.

Example 4 Group E The farm game

A farmer has a lot of mice, guinea-pigs and hamsters. He is very kind and lets any children play with these animals as long as they keep to the rules of the farm. The rules are:

Rule 1. Any child is allowed to ask for one, two or three animals, but only for one of each kind. Or he can just go and watch the other children play, and not have any animals.

Rule 2. You first play in a square room, but as soon as there are two children in that room who *between them* have asked for a mouse, guinea-pig and a hamster they must both leave the square room and go and play in the round room which is next door to the square one.

(For example if John has a guinea-pig and a mouse, and Susan has a hamster, then John and Susan must go into the round room)

Rule 3. As the animals get hungry, they must be given back to the farmer for feeding and other animals can be taken in exchange. The farmer has a code to let the children know which child can have what kind of animals in exchange for the hungry ones. He projects either one or two or all three kinds of animals as one picture on a screen, but only one of each kind. Each child must ask for one of each of those animals that are either in his hands or on the screen, but not in both places.

(For example, if a hamster-mouse child sees a mouse on the screen, he asks for a hamster. A 'no-animal' child asks for whatever is on the screen. A mouse-guinea-pig child on seeing a guinea-pig and a hamster on the screen asks for a mouse and a hamster, etc., etc.)

Rule 4. If the farmer projects more than one picture on the screen *at the same time*, then each child can ask a friend in for each extra picture that is on the screen, and these friends can have the extra lots of animals which the code tells them they can play with.

(For example, suppose the farmer projects five pictures:
one M-G picture, two M-H pictures and two M-H-G pictures.

What does an H-G child do? He asked 4 friends to come in, and they will become

one M-H child, two M-G children and two M children.

What happens to an H child? He and his 4 friends will become

one H-M-G child, two M children and two M-G children.

There are not yet any children who must go to the round room)

IS IT POSSIBLE FOR THE FARMER TO SEND ALL THE CHILDREN TO THE ROUND ROOM?

The mathematical content of this game is a four-dimensional algebra. Other games whose contents were the complex algebra and other two- and three-dimensional algebras had already been constructed by these children, with 'cover stories' and without. They were therefore already somewhat familiar with the transition from the story to the structure of the story, leading to the manipulation of the rules inherent in the story. Certainly the initial interest was aroused by the imagery created by the story. The animals, the children playing with them, the animals getting hungry and needing to be fed were all situations which aroused considerable affective response, and you could hear a pin drop while the story was told. It was not long, however, before the intricacy of the rule-structure took over as the chief motivating factor. For example, three of the girls became very involved in studying the mathematical curiosity that you could have two non-zero vectors whose product was zero (i.e. that it was possible to send all the children to the round room without putting 'cancelling' pictures on the screen). They spent some time in accumulating examples of such vectors and wanted to find 'the pattern', as they called it, of what made this possible. It was finally possible to persuade them to restrict themselves to a simpler variant of the story in which there were only two kinds of animal. This resulted in a two-dimensional vector space (one dimension being the no-animal and the two-animal children, the second dimension being the one-animal children). The problem in four dimensions required the knowledge of the properties of 4 by 4 determinants, and correspondingly even in two dimensions the knowledge of linear dependence and its relationship to the vanishing or non-vanishing of 2 by 2 determinants was technically required for a complete solution of the problem. To enable the children to succeed, a vector which would act as zero

divisor (i.e. which could be multiplied by another non-zero vector and yield a zero product) was suggested. From this they found all possible zero divisors in the algebra embodied in the two-animal story. They could not prove that there were no others although they seemed reasonably sure that there would not be any more. They were happy to learn that their conjecture was correct but less pleased to be told that they did not yet know enough mathematics to prove their conjecture.

During the course of their investigations these girls developed a symbolism, using the first letters of the animals. At first they drew plans of the farm, but later they were quite satisfied with notation such as

$$(N+H)(N+G)=N+HG+H+G=0$$

which they evolved with practically no help. The reason they gave for developing the notation was that they could not remember all the cases which sent all the children to the round room so they had to find some way of writing these down, otherwise how could they ever find the pattern?

This shows that it is possible to use representational games as a starting point for activities that will lead to quite high-order mathematical thinking, provided the game is suitably structured. Representational play soon went over into rule-bound play in this instance and eventually the properties of the rule-structure were what provided the interest. Having explored the working of the rules, i.e. having played the game by the rules, the children's way out of the closure was a serious and determined attempt at analysis. This is the way in which the rule-bound game is reopened, enabling the player to ask questions *about* the game! The children could look far enough ahead to realize that they were looking for a rule about the rules, i.e. for the rule covering zero divisors. They had their eye on the eventual closing of the next cycle.

Here is another example, illustrating the same point.

Example 5 Group A Martian rockets story

Some rockets are going to Mars; on each rocket there are as many motors as the total number of rockets that are going. During the flight one rocket hits a meteorite and is destroyed. Altogether 6 other motors on the other rockets get put out of action, but all these reach Mars. On

Mars they find some empty rocket ships left there by some other space-travellers, but these have no motors. They remove some of the motors from their own space-craft and fit the Martian rockets up with them. They find that it is just possible to have the same number of motors on each ship, the Earth ships as well as the Martian ones. They manage to get home to Earth on reduced power, taking the Martian ships with them.

How many motors were they short of on each rocket ship on the return journey, and how many rocket ships did they find on Mars?

Can you make up a problem like this one?

(Upon arrival *Subject* immediately found the ruby-headed eraser)

Subject. 'I'm going to report you to the burglars' association.' (He read the problem) 'Let's choose a number for the rockets. Ughlaglagha. That's the Martian word for number.' (S checked the spelling and crossed out the last h. *Carol* came over and took the eraser)

S. 'I'll have plenty of these [rubies] when I go to India.'

Observer. 'When is that?'

S. 'In September.' (He read the problem aloud, using the word ughlaglaga) 'How many motors, it doesn't say?'

Dienes. 'That's the problem.'

S. 'I don't get it.'

D. 'It doesn't matter.'

S. 'Then ughlaglaga. I'll do it with a number. 5 rockets. Then there are 5 motors in each rocket. 6 motors get put out of action. How many motors are left?'

D. 'You can't find out. You can find out how many are missing.' (D moved away. S leant back in the chair) 'I don't get it, I wish they'd give a number. I think I'll do some burglarizing.' (S walked around the room and then came back. O read the problem to him)

O. 'How many motors were missing when they reached Mars?' (S was able to say, but he wanted a 'hint' as to how many Martian rockets there were)

S. 'I think I'll do some more burglarizing.' (D came over)

D. 'Let's have the washers as the motors. Then there are 5 washers and 5 orange ships.'

S. 'We'll call this [the balance] the meteorite. It's a huge meteorite.' (S picked up a rocket ship and had it collide with the balance-beam .He then had several other rockets collide, losing several motors on the way. One ship lost 2 motors, one lost 4 motors, and two didn't lose any) 'This one that doesn't lose any is going to save the space-man' (in the rocket that has been destroyed). 'How many did I lose? I lost ughlaglaga+6. I'm going to experiment. They're experimenting with the Martian rockets.' (He put motors on all the Martian rockets) 'There aren't enough. They saved one rocket.'

O. 'No, they all burned up, but you want to bring back the earth-rockets, too.' (S was using only the 5 Martian rockets)

S. 'I have to put 2 motors each on the Martian rockets and one on the earth rockets, 'cause the Martian rockets are bigger.'

O. 'No, they should all have the same number of motors. Maybe that isn't the right number of Martian ships.' (S took one away, and then another)

S. 'They both hit a meteorite.' (S kept on dropping washers on the floor, so it was difficult to keep track of the right number of rockets) 'Maybe there's only one Martian rocket. No, I'm going down to zero. No, that doesn't work.' (He tried 2 and then 3) 'They captured 3 if they started with 5.'

D. 'What if they started with 6? Would it still be 3?'

S. 'I don't know till I try it. There are only 5 orange rocket ships.'

D. 'That's O.K. since one gets lost almost immediately anyway.'

S. 'No, I use a Martian rocket at first. It'll turn into a Martian rocket later. Aren't I clever?' (He again brought the rockets one by one to Mars)

D. 'Now is it still 3 rockets?'

S. 'I don't know.'

D. 'Do you think there are 4 now?'

S. 'I don't know.' (He took one away, and put 3 motors on each of the rockets)

D. 'There are 3, aren't there? So there are 3 engines missing both times, and 3 less motors again. Would it still work with 7?'

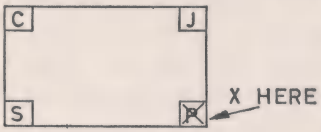
S. 'Yes.'

D. 'Are you sure?'

S. 'Pretty sure, because it works for an odd and an even number.'

Carol. 'Where's my ruby-headed eraser?'

S. 'I'll draw a treasure-map.' (On the board he drew)



S. (to O, pointing at the X spot on the map) 'That's where it is supposedly.' (Just before leaving, S showed me where it really was. Behind the typewriter case under his own desk, and not under R's after all) 'Don't touch it.' (He left)

The subject in this report cannot be said to have reached a firm mathematical generalization on the factorization of the quadratic function x^2-x-6 embodied in the cover story. The sessions were at this time drawing to a close, the children in the group had been stretched very nearly to the limits of their mathematical capacity, and it is quite likely that without the cover story no mathematics would have been learned at all by this subject at the session just quoted. The framework of the story was just loose enough to generate enough zest for him to play along with the experimenter, yet structured enough to direct the activity, at times almost uncontrolled, into mathematical channels. It may be that the extent to which a game is structured will turn out to be an important variable which can be manipulated to the educational advantage of students.

2.4. *Analytical insight through play*

So far we have looked at play as one possible vehicle which could promote the construction of mathematical concepts. With suitable structuring it could equally well be used for the analysis of what has already been constructed. The constructive work that may be achieved through planned rule-bound play has been illustrated in the three cycles described on pages 32–35. The predicate in one cycle becomes the subject in the next cycle, on which a new predicate operates in the new cycle, and so through the superposition of these cycles mathematical structures are built up. It is possible to go quite a long way in building up these structures[1] without becoming aware of the relationships

[1] See Z. P. Dienes. *Building Up Mathematics* (Hutchinson, 1960).

between or within their parts. It may be conjectured that, if it is possible to design mathematically constructive games, it should be equally possible to design mathematically analytical games which aim at relationships between structures and parts of structures. One example of such designing is the construction of algebras out of groups and vector spaces. This was done with Groups D and E.

The idea of a 'good game' is developed. A game, first of all, consists of states and transitions between states. The transitions between states are called moves. A game is a good game if:

(1) It contains a 'no move', i.e. no change of state.
(2) A succession of any two moves can always be replaced by a single move yielding exactly the same result as the two moves replaced.
(3) For every move there is a corresponding unique inverse move, which, if performed after the first move, restores the original state.
(4) The game is associative; i.e. if A, B, C are any three moves, A followed by B can be replaced by Z, B followed by C can be replaced by X, then Z followed by C has the same effect as A followed by X.

The last rule becomes superfluous if we restrict ourselves to games which are transformations of states, as all transformations are associative. A number of games can be played, some good, some not, which can be followed by the construction of vector spaces. An n-dimensional vector space, together with a 2n-move game, would define an 'algebra'. For example the following 4-move game, together with a two-dimensional space, will yield a two-dimensional algebra.[1] This was done with Groups D and E.

Take a piece of cardboard with an arrow on it. Let us say that the arrow can be in one of the following four positions, which will be the states in the game:

↑ ← ↓ →

We can have four moves:

N = No move, L = rt. angle turn to left,
T = 180° turn, R = rt. angle turn to right.

[1] See Z. P. Dienes. *The Power of Mathematics* (Hutchinson, 1963).

It is easily verified that this is a 'good game', and the rules are

NN=N,	NL=L,	NT=T,	NR=R,
LN=L,	LL=T,	LT=R,	LR=N,
RN=R,	RL=N,	RT=L,	RR=T,
TN=T,	TL=R,	TT=N,	TR=L.

To introduce 'vectors' we can take *sets of arrows* in the permitted positions. Arrows pointing in opposite directions cancel out.

Now in order to introduce a multiplication, the states can be related to the moves by choosing a move to be associated with each state. For example

↑ could be associated to No move, ← to the left move,
↓ to the move turning things round 180°, → to the right move.

According to this

the vector (↑ ↑ ↑ , →) multiplied by ← would be
(← ← ← , ↑) since everything has to be turned left.

Multiplying by a compound vector is defined by postulating the distributive law of multiplication. For example

(↑ ↑ , ←) multiplied by (↓ , →) would be (↓ ↓ , →) as well as (→ →, ↑); and adding these we have (↓ , → → →) which is the product vector. In this way we have constructed an isomorph of the complex algebra, where for example ↑ ⇔ 1, ← ⇔ i, → ⇔ − i, and ↓ ⇔ − 1.

The algebra includes both the vector space and the 4-move game. We obtain the 4-move game as a particular case of the algebra if we restrict ourselves to the vectors

↑ , →, ↓ and ←

and use the rules of multiplication that hold in the algebra. We obtain the vector space out of the algebra if we do not do any multiplications. Both the vector space and the 4-move game are *particular cases* of the algebra, and so the algebra is a *generalization* of both as it is an extension of both. These are some analytical considerations which can be led up to through this type of play.

It is of course highly improbable that any children who have played the vector-space games as well as the group games will spontaneously invent the possibility of combining them into algebras. It would be too much to expect children to travel hundreds of years of mathematical development unaided. On the other hand, once children are given the idea they are more likely to construct algebras of their own. It will now be seen that the mouse-hamster-guinea-pig farm game is an isomorph of a four-dimensional algebra built on the four-dimensional vector space whose dimensions are represented in the story by

N & MHG; M & HG; H & MG; G & HM;

as opposites cancel out and must go to the round room. Any collection of children playing in the square room at any given moment is a vector. All possible vectors form the vector space. The multiplication is provided by an 8-move game,[1] which is given by the rules for exchanging animals when they are hungry. The algebra is constructed by the multiplication of the vectors by each other, where one vector is the group of children in the square room, the other one the picture of the animals projected on the screen by the farmer.

Children in both Groups D and E constructed the complex algebra. Group D had the cover story first, followed later by manipulation of vectors on tables, each manipulation being equivalent to multiplication by a unit vector in the algebra. Only one child in Group D saw that these manipulations were the same as the 'magic' in the cover stories, whereas in Group E it was generally realized that the story and the manipulations were 'really the same'. The reason for this might have been that Group E had longer experience than Group D, or perhaps that Group E were given more freedom of choice and so more energy was available, or that Group E consisted of children who were two years older than those in Group D.

2.5. *Play as interference with learning*

We should not imagine that play is always a harbinger of more advanced thinking activity. In many cases play is resorted to as a kind of defence mechanism when the cognitive going gets too tough for the learner. Examples of such defence mechanism type of play will be given in the sequel. The affectively loaded, projective type of play appears to

[1] See game No. 7, Appendix I.

be of least immediate use to mathematics-learning, although it may well be very necessary as an outlet for the subject. Interfering play may also result from an overwhelming need to use the imagination actively. It is just as impossible for a person with a very fertile imagination not to be imaginative as it is for a very muscular person not to want to use his strength, or an intellectually endowed person not to use his brain-power. We shall give a number of examples where imagination obtrudes into rather than helps the mathematical work, or at best is neutral in the mathematical activity. Such intrusions usually replace mathematical activity by some other activity where the imagination has freer play to express itself. It is perfectly true that to be a good mathematician it is necessary to have a good imagination and be well supplied with imagery, but this imagery must be disciplined in no uncertain manner before it will bear mathematical fruit. This is equally true of any other kind of expression of imagination in external form, such as painting or music: discipline is essential for any such purposeful activity. It is undisciplined imagination, giving rise to a flood of uncontrolled imagery, which is likely to get in the way of mathematical thinking.

Example 6 Group A Faces

D. 'What are you working on today?'

S. 'Faces. I'm going to make a bunny rabbit.' (Erased as soon as it was done) (To D) 'I'm studying faces. That's my father when my mother screams at him . . . I'm going to make a cranky old witch.' (She started, but erased it quickly because D was about to take a photo) . . . C, why don't you make faces? . . . My father—when mother screams at him. My father's really a grouch—really—so I'll make him after a dinner—a really good dinner—he's almost asleep.

This is mother—at the beach. She's wearing sunglasses, and a new-style sunhat.

The sun's almost ready to burn up her hat.'

(S added the flames to the drawing she had already made)

'Oh, oh—her hat's on fire. There are flames on her head.

I'll make it into a Jamaica lady—carrying a bundle on her head. She's wearing glasses. (S made the flames into a bundle) I've got to make the Jamaica lady's face purple.'

(S used purple chalk to do this. She began rough-housing with C)

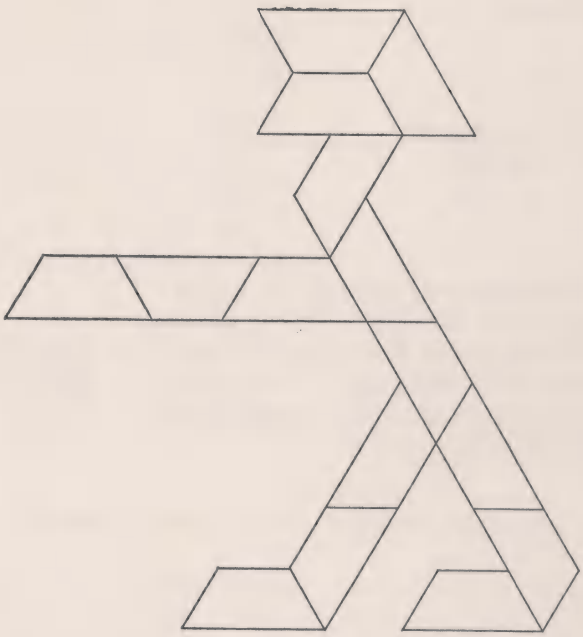
Example 7 Group A Reaction from formalism

S. 'I'm making an N. I am doing all three groups in one large group.' (Then C came over and provided a momentary distraction. S appeared to be blocked. He said he felt like building something with the squares and the triangles) 'I thought it was dumb and cuckoo just to build something, but perhaps I should.'

9.20. S said he needed 360 triangles to build what he wanted because it would not work otherwise. He then went to the box and took some small triangles.

Example 8 Group A Building a knight

S. 'Let's see, four, three, two blocks, now I'll try six.' (He thinks) 'You know what? I made knights out of the boats in the classroom.' (S had just been told that he could build things out of blocks any way he liked, and at the same time the girls were joking loudly at the blackboard. S gets the boats from the table, builds his knight to show to O. He makes it out of fifteen trapezoids, shown on figure over page)



Example 9 Group A Rattlesnakes

S. 'Have you ever met an ambassador's son?'
S. 'How many pegs?' (S chose several very large numbers)
O. 'There won't be enough cups for those numbers.'
S. 'I want a drink of water.' (He took one of his cups and left the room. He returned with water in the cup. S took 4 cups from the pile and counted the remaining cups) 'There are 34; I need another drink of water.' (Again S chose some very large numbers) 'No, I'll put just 4 pegs in the cup.' (By mistake he put 4 in the wet cup, then he put 4 in another cup, and put that cup in the middle of the stack of cups. S did this several times, so that the cups were well shuffled. He continued putting 4 pegs in a cup and then putting that cup somewhere in the stack)

S. 'Don't peek, close your eyes.' (He continued hiding cups in the stack) 'I'm trying to make a leaning tower of Pisa.' (When he finished putting 4 pegs in each cup, S returned to the card)

S. 'I don't get it.' (He took 4 cups. He checked the tower to 'make sure there are no empty places')

S. 'Look, it's the last 2. It looks like a snake, it's a rattlesnake.' (He showed it to C)

D. 'Can we borrow some cups?'

S. 'No, my number was 34, I can't pick a smaller one. Look, what's the name of the musical instrument?' (He'd placed his cups like an accordion)

O. 'It looks like an accordion.' (Some water fell out of the cups while S was holding them sideways. S drank the water, spluttering over the pegs)

S. 'Rattlesnakes away!' (He aimed the snake at me, it fell on the floor. *Sam* tried to get S interested in some numbers on the blackboard. S just wanted to play with the snake)

Comments for the day

S wasn't at all interested in working today. He arrived late because of the snow, and then continued playing throughout the session.

Below we give three examples from the protocols of the same subject as the one reported in Example 6, illustrating the intrusion of elements of imagery into the mathematical activity. This subject found it very difficult, indeed pointless, to look for regularities in the inductively oriented series of tasks in which she was engaged. This appeared to her as an adult-imposed activity. She would often oblige with a rule, but would be unwilling to try the rule out in other situations to test its validity as she probably knew quite well that it would not work. The individual 'hair-styles' on the accompanying peopled triangle give an indication that apart from being dominated to a large extent by her fertile and untamed imagination she very much looked at the world as made up of individual things rather than of a set of regularly occurring patterns.[1] She was in fact the least successful in Group A in coming to grips with any mathematics at all.

[1] This was Sam Anderson's diagnosis.

Example 10 Group A Hair-style triangle

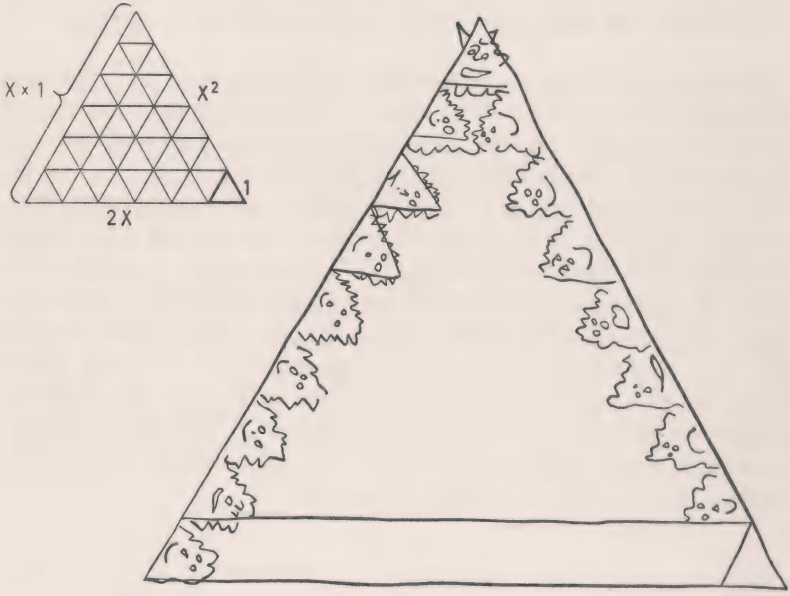
. . . Finally D has S draw the triangle again, i.e. the 64-triangle figure, including the new row. When she has completed this, she draws wavy lines round the borders of the triangles to their bases along two sides, calling them 'hair-styles', and then proceeds to draw faces inside these triangles. She puts faces in just those triangles that she intends to count; she was going to discover the total number without counting them all. It is of interest to note that she included a face on the new row only on one side; perhaps she was unsure of the unit small triangle which was to be added as a final step after the number of triangles in the final row without it had been computed.

The drawing made by the subject is below.

The task was directed towards the subject's eventual realization of the relationship

$$X^2+2X+1=(X+1)^2$$

by structuring the triangles as set out below



Example 11 Group A Harvard-Yale game

S. 'I have to use elastics all the time.' (She put 5 rings on hook 5) 'I'm 5. No, I'm 21, and I've just learned to vote.' (S noticed a mis-spelling on the card. The word THINGS was spelled TINGS. She told D there was no such word as TINGS. He suggested she change it and she did so. S exhibited much playfulness here; she squealed, made funny faces, etc.) 'Harvard's defeating Yale.'

O. 'How's that?'

S. 'This is the Yale side.' (She pointed to the left side of the balance) 'And this is the Harvard side.' (The right side) 'But Harvard's on both sides.' (She had used red elastics on both sides, and had not used the blue elastic which referred to the Yale team)

D. 'Can you do it another way?'

(S took the balance beam apart. C took her washers)

S. 'Please give me back my poor things. They're afraid.' (Holding the balance beam up in the air) 'It's a Christmas tree, look.'

O. 'Did it grow that way?'

S. 'Yes, it's lopsided. The washers fell off. . . . Oh, my Christmas-tree ball broke.'

Example 12 Group A Parking lots

(S next read a note on the wall about rules for cars parking in the parking lot at school. She immediately made a game out of this)

S. 'The blocks are cars. The larger figures are the parking lots.' (She lost interest in this and closed her notebook and put it carefully away in the corner of her desk)

S. 'I can't look. I have to make guesses without looking at it.'

S. '9 is one half of 18, 18 is one half of 36. That's 4×9. 2 high, 2 wide, 9 long, but I don't want to make it.'

In the last two examples particularly the imagery came bursting in quite compulsively. For this subject every situation is really the situation plus the little bit that she adds to it from her imagination. In the

parking lot game there appeared to be no interference with the final piece of mathematical thinking, as she produced a correct solution and she was so sure that it was correct that she did not want to make it.

It is problematical whether the kind of imagination displayed in the Harvard-Yale game and the Christmas tree can be effectively harnessed in the service of a systematic activity such as mathematics-learning. On the other hand repression of such expression would probably have a generally repressive effect which the subject is likely to apply to all things mathematical. One wonders to what extent such repressions are the main cause of mathematical-learning blocks. In the Harvard-Yale game S was letting her imagination take over entirely. The balance became the Harvard-Yale game, then it became the Christmas tree; the washers not only became Christmas-tree balls, but were animated, as indicated by S's exclamation, 'Please give me back my poor things, they're afraid!' All this was very much the primary type of representational play, taking up S's entire attention and not only interfering with, but entirely displacing any more organized cognitive activity. If such untamed imagination is not to be suppressed, ways and means must be found of directing the underlying energy into more organized patterns.

2.6. *Summary*

We have given a number of examples to illustrate ways in which play-energy may be transformed into higher-order cognitive activity, as well as militate against the inception or continuance of such activity. What the laws are that govern such transformations we do not profess to know, but that such transformations can and do take place is plainly the case. Finding ways in which the sometimes explosive character of play-energy can be converted into a more evenly functioning stream so that it may be used in the service of learning is a problem that exists and urgently needs solution.

There are some pointers, however. When play is the acting out of of some emotional disturbance it is not likely to lead to immediate learning, although without such acting out future learning may become more difficult. There also seems to be a tendency for manipulative play to lead to concept construction and for rule-bound play to lead to analysis, in particular to generalization.

We have seen that rule-bound play has considerable auto-motivating

value. The excitement of getting to grips with a rule-structure appears to provide the motivation for going on with the enquiries, provided there has not been too much previous punishment-reward conditioning. For example, there was a marked difference between the Groups D and E in the extent to which they were motivated by the tasks themselves. In fact Group D had at times to be offered extrinsic rewards to keep up the flow of interest, although task-centred motivation increased as the sessions proceeded and the children became intrigued with the work. In Group E the level of interest was constantly high; work was discussed out of school, the children asked for homework, and there was great disappointment when the sessions ended. Group D were taken from a punishment-reward-conditioned public (i.e. state) school; Group E from an independent school where special emphasis was laid on self-reliance and grades were little used for motivating children to learn. One might say that the mathematical game somehow captures children's diffuse play-energy and creates a general atmosphere of excitement and enquiry favourable to learning.

The transition from play (in the technical, mathematical sense) to learning mathematical abstractions implies cutting through a certain amount of noise in the games; irrelevant noise is progressively discarded and a more and more mathematically pure essence distilled. Noise can be artificially reduced by restricting the children's freedom of action. This cuts out the noise, but with it also some of the fun. It is an open question whether the fun and playfulness introduced by loosening the structure is helpful or harmful to the learning of mathematics. Play may recruit zest and so increase the energy available, but it also creates noise, and cutting through the noise may use up more energy than has been created by the release of the play-energy. The problem appears to be to bring children face to face with structure; once they have tasted the excitement of coping with a structure and investigating how it works, playfulness will show itself rather as delight in coping than in overt play.

There are a great many problems that remain. It would be interesting to do some studies in which the amount of choice given to the children was varied. It is possible to start on a mathematical road and leave everything to initiative and experiment or to programme all choice out of the way. In between these extremes there are a great many ways in which the amount of choice may be controlled. It is not at all obvious

where the optimum situation lies, or even if there is an optimum situation uninfluenced by other variables. Nor is it clear what the criteria for effective learning should be. There are questions about the best order in which representational and rule-bound games may be played for most efficient subsequent mathematical learning. There is the problem of the relationship between the extent of manipulative play and eventual mastery of the rules. There is finally probably the most intractable problem of measuring the noise present in any game in relation to any given mathematical structure, and the relationship between the amount of noise and effective learning. More intelligent children are able to cut through more noise, but less intelligent children need noisier situations and more of them to abstract mathematical concepts. Perhaps the solution lies in gradually increasing or decreasing the amount of noise present. Varying noise deliberately as part of the process of education will only become possible if and when enough is known about its place among all the other variables relevant to learning.

2

Abstraction and construction

3. *Some theoretical remarks*

In the last section the transition between play and the higher cognitive processes was discussed. In this section I intend to examine the functioning of these processes themselves. Before going into the processes in any detail, I must first make clear in exactly what sense I shall be using such terms as abstraction, concretization, generalization, particularization (or restriction), construction, analysis, symbolization and interpretation. Some of these problems have already been discussed rather more theoretically in a paper.[1]

3.1. *Abstraction*

This is the extraction of what is common to a number of different situations and the discarding of what we choose to regard as 'noise' or irrelevancy. What will be regarded as noise is quite arbitrary and is determined by our aims. In the same situation we might regard a certain attribute as relevant when we had one aim, and irrelevant when we had another aim.

According to this view of abstraction it is just another word for the formation of a class, the end-point being the realization of the attribute or attributes which make elements eligible or not for membership of the class. In this sense every word denoting a 'kind of thing' indicates a class, and nearly every word does just that. The word 'chair' calls to mind the class of objects known by that word. The attributes that an object necessarily possesses before it is called a chair are not always appreciated consciously, and this is where we find one of the differences between mathematical and ordinary thinking. The class 'chair' is quite operational enough for us to recognize an enormous variety of objects

[1] Dienes, Z. P. 'On abstraction and generalization', *Harvard Education Review* (Summer, 1961).

under that category without always being able to give a necessary and sufficient condition for an object to be a chair. As we move further towards mathematics, this gets less and less possible. *Awareness is almost indispensable for a mathematical class to become operational.*

In a *mathematical* abstraction we are classifying some rather complex 'objects', i.e. objects of our thought. It then becomes somewhat difficult to get the 'feel' of these objects by practice (as we can with chairs) and adjudicate correctly every time whether a certain complex structure is like a certain other complex structure from a particular point of view. True, mathematicians have hunches, and the good ones are those whose hunches turn out to be correct more often than not. But no mathematician worthy of the name would leave a hunch a hunch. If he could not be bothered with it himself, he would farm it out to a colleague or to a research student. The identity of structure has to be explicitly shown. It is not obvious that a superimposition of two transformations of the kind

$$\begin{aligned} X &= ax + by \\ Y &= -bx + ay \end{aligned} \qquad \text{and} \qquad \begin{aligned} u &= cX + dY \\ v &= -dX + cY \end{aligned}$$

will combine in exactly the same way as the complex multiplication

$$(a + bi) \times (c + di) \qquad \text{where } (i^2 = -1)$$

Someone might have a hunch that this was so, but it would have to be shown. If we carry out the substitution in the first sets of transformations, we obtain

$$\begin{aligned} u &= (ac - bd)x + (bc + ad)y \\ v &= -(bc + ad)x + (ac - bd)y \end{aligned}$$

and if we carry out the complex multiplication, we obtain

$$(ac - bd) + (bc + ad)i$$

Then if we make the following correspondence:

Top left-hand coefficient in matrix	corresponds to 'real' part
Top right-hand coefficient in matrix	corresponds to 'imaginary' part

we can 'translate' one activity into the other. They have the same pro-

perties as far as the interrelations of their properties are concerned. We have identified two mathematical structures that really look rather different, and found them to have the same structure from a certain point of view. The interpretations of $i^2 = -1$, the transformations of spaces and so on are for our purpose here sheer noise. If we discard this noise and concentrate on what is common, we are on the way to realizing the set of attributes which a structure must possess before we can describe it as 'the complex algebra'. This is the mathematical abstraction process and this is what is imitated in the classroom when identical structures are provided, so that what is common should stand out and noise should be recognized as noise and discarded.

Abstraction here is essentially the formation of an isomorphism. Isomorphism means more than a one-one correspondence; it means in addition that, as far as at least one operation is concerned,

> if *operating on A by B results in C* in one situation,
>
> then *operating on the isomorph of A by the isomorph of B should result in the isomorph of C* in the other situation.

If this is so for all possible ways of operating, and is so *both ways*, then we say that the two situations are

isomorphic under our operation.

In other words the correspondence must be quite flawless. One single exception would be enough to invalidate the isomorphism. The formation of isomorphisms is the process by which we arrive at our abstractions. These abstractions will become the compounds of the properties defining the class co-extensive with the abstraction, whose elements are all possible structures that have these particular compounds of properties. Forming an abstraction in some sense 'clears up the mess' and we know where we are. We can handle the new class as a possible element in a further as yet unformed class, and the process begins again. The forming of an abstraction is an end-point in a psychological as well as in a mathematical cycle.

It is the explicit formation of an isomorphism that puts a mathematical class of entities into a somewhat more precise category than an everyday class. This is a very powerful tool by which means we can create order from a great deal of mathematical mess in a remarkably effective way.

3.2. *Concretization*

This process is the reverse of abstraction. Very often we need to pass from the class of 'such and such structures' to a concrete example of such and such a structure. This is particularly important if the class has not been formed by construction from its elements but 'from above', so to speak, by logical combinations of attributes. In these cases existence theorems may become necessary. There are two good reasons why we may want to prove an existence theorem. One is that we may wonder whether we have overdone things; whether perhaps the particular combination of attributes under consideration is essentially impossible on account of the very nature of the attributes themselves. In such a case it is enough to decide whether the attributes in question are in fact incompatible or not; i.e. whether there is any logical reason to believe that the things do not exist. The other reason is that we might simply be curious about what the creatures we are talking about really look or feel like. The solution by non-contradiction will not satisfy us if we are curious to have a look at the elements of our class. If the class has been constructed by a logical combination of some other classes we have no idea of the kinds of things that we are supposed to be talking about, and we might want to have a look at some specimens. We might have a theory about the kinds of metabolisms we might or might not find on the planet Mars, but surely if we are really curious it is more satisfactory to go there and find out?

To sum up: abstraction is to proceed from elements to class, concretization is to proceed from class to elements. Isomorphisms help us to be sure what the elements are that belong to a certain class.

It is important to add here that forming an abstract idea in the sense of forming a class is really a psychologically irreversible concept. As Jerome Bruner has often said, it is impossible to recapture pre-conceptual innocence. We can go back to the elements after we have formed the class, but things will never be quite the same as before. Concretization may help in planning the path for future constructors of the class in question, but it will not reverse the process for those who have constructed it already. By contrast we shall see that the process of *generalization* can be reversed, becoming restriction or particularization, as here we are dealing with classes already formed, passing from smaller to larger classes when we generalize or extend, and passing from larger to smaller classes when we particularize or restrict.

3.3. *Other processes in mathematical thinking*

Before going on to examine the abstraction process in more detail, I shall sketch a number of other typically mathematical processes, in order to view the abstraction process more in context.

Generalization. In the most primitive sense this is becoming aware of a class-inclusion situation. It means that we have discovered that if x belongs to class A, then it also invariably belongs to class B. We say that class A is included in class B, or that the class A can be extended or generalized into class B. In the vector-space game a one-dimensional vector space is included in a two-dimensional vector space. If A is the cups-space, and B is the cups- and boxes-space, then clearly any vector with cups only is also a vector with cups and boxes, except that there do not happen to be any boxes in it if it belongs to the cups-space. For the more mathematical generalization to take place, abstraction of some kind must take place first.

Restriction. In mathematics we often deal with classes, which we conjunctively join with other classes. In other words attribute A *and* attribute B have to apply to the new class. This is clearly a restrictive practice and proceeds in the opposite direction from generalization. We might say that we can only have vectors in which there are always twice as many boxes as cups, and say closed boxes must occur always with upside-down cups, open boxes with right-way-up ones. We have now proceeded from a two-dimensional space to a one-dimensional space, because in our restricted class all vectors are multiples of

2 closed boxes + 1 upside-down cup

or

2 open boxes + 1 right-way-up cup.

This is just as it is in the cups-space; every vector is a (positive or negative) multiple of one cup. We can *restrict* by decreasing the number of dimensions in a space (each linear condition takes a dimension off); we can *generalize* by adding more dimensions. This is of course just one simple example.

Construction. This is literally putting things together to build another structure with some previously specified requirements. The re-

quirements may be more or less stringent, more or less specific. In Bartlett's terms they are the 'standard' towards which the construction is aimed. The accent during constructive activity is on the *kind of thing* the end-result is going to be. It cannot be the actual thing, as it would then be constructed already. Of course for this kind of activity to have much chance of success the classes that are used for the construction must be well formed. By this I mean that any given exemplar for which the attributes defining the class are relevant or meaningful (i.e. any element in the universe of discourse) will be recognizable as belonging to the class or as not belonging to the class. This is not quite necessary psychologically for constructive activity to take place, but it is probably necessary for it to take place efficiently, that is, with a good chance of reaching the end-point.

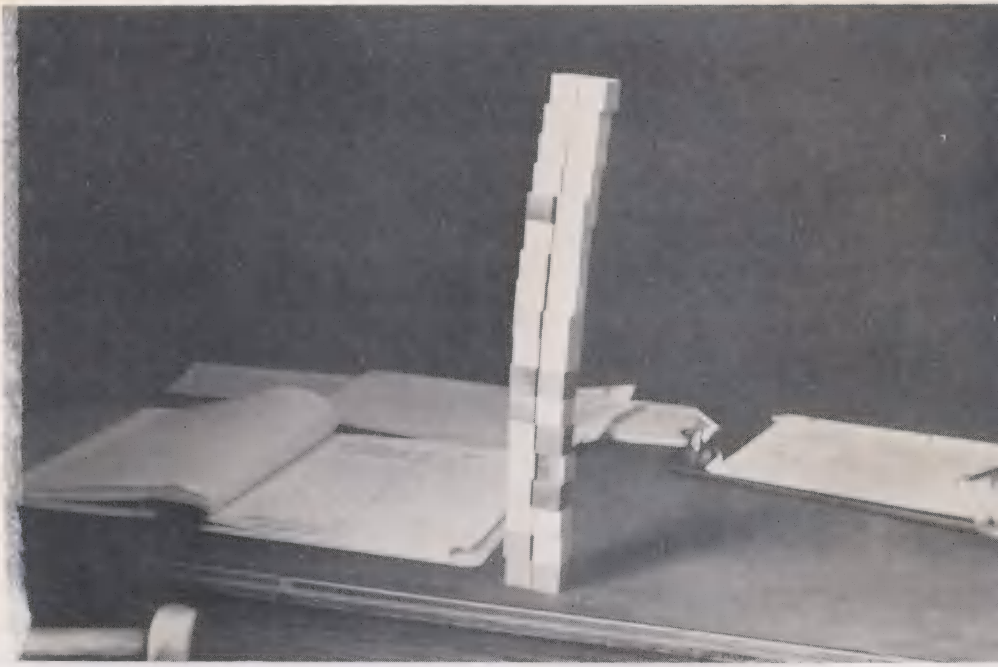
Analysis. This arises both out of construction and out of rule-bound play, but mostly out of the second. When a certain number of games have been played, a certain amount of classificatory thinking will gradually be forced upon the player. Is this kind of game always also that kind? If it is this kind, can it never be that kind? Attributes are coined from the manipulatory activity (class-formation), and questions are asked about the relationship of these attributes after their regularities have been explored in rule-bound play. This is essentially analytical activity. The really 'meaty' part of analysis consists in convincing ourselves fully that the attributes do in fact define classes which are in such and such relation to each other. This is called 'proving a theorem': the suspected relationships are the theorem-to-be, and convincing ourselves is the proving of it.

In a way, analysis maps out the mathematical country. The theorems are warnings about what you may find if you take a certain road. New theorems are established by looking at the unknown relationship in relation to the known relationships; the latter relationships form our bridgehead for an attack on the former.

We must not lose sight of applications. This is essentially an attempt to find a 'best fit' between what we see around us, and the mathematical structures we have internalized. Such activity has a stimulating effect on the other ingredients described, at any rate with some people. For some mathematicians it is very important to remain aware of the thread of reality which reaches into the most abstruse mathematical theories,

The constructive stage of finding factors—6×6

The constructive stage of finding factors—2×18



and applications are stimulating because they reinforce this awareness. Others prefer to ignore the 'reality component' of mathematical theories and find a thrill in their ability to manipulate an extremely complicated game. Both types have provided mathematics with extremely valuable contributions.

Symbolization. A symbol is a sign that reminds us of something apart from the sign itself. A word written down does not only call to our attention the squiggles of the actual letters—in fact these are only consciously noticed when someone is learning to read. The word-sign calls to our attention a certain imagery which accompanies the class of things that the words symbolize. This imagery will vary with the context in which the word is read; for example the word *tree* in a poem about a tree would probably evoke quite a different image from the same word in a treatise on Botany. When a mathematician 'thinks' he usually does so to the accompaniment of imagery of one kind or another. He cannot, of course, rely on these images, and in any case the images are restricted by the 'rules of the game', but even so he might think of quite inappropriate images which would lead him to incorrect hunches, so he has to have a control over his thought processes. For this he uses symbols, the rules for manipulating them embodying the rules operating within the structure that he is investigating. Without such symbols he could not possibly keep together the enormous amount of information which might have to be considered simultaneously in order to arrive at a new connexion. Symbols are a kind of primitive computing machine that will do some of the thinking for us, and so make for great cognitive economy.

In the free-for-all of mathematical research the mathematician will invent his symbols when he feels he needs them. This is a natural situation, and one might conjecture, although this is by no means obvious, that it is the optimum situation. Just as we tend to eat when we are hungry and rest when we are tired, unless prevented, we might assume that we go to the symbol stack when we are in need of symbolic refreshment. This situation does not of course obtain in schools. If it is appropriate to introduce symbolism at certain points and less so at other points, then the wholesale introduction of everything at once to all children is likely to cause wholesale conceptual carnage in their minds, which is in fact just what happens. (This does not mean, of course, that this explanation is the only one, nor indeed that it is a true one)

The problem of how and when to symbolize remains with us, and its clarification may be necessary for much of the work that needs to be done in the study of mathematics-learning.

Interpretation. This is of course the reverse of symbolization. The symbolism used by a mathematician, or by anyone, is a kind of code that needs de-coding. When we listen to someone speaking we are doing just this kind of de-coding all the time, and doing it extremely rapidly. We decode mathematics less rapidly, probably because the symbols have not been as firmly fixed to what they symbolize, and also for lack of the prolonged practice that we have in our own language.

There is one difference between a conversation and a mathematical development. If we have not really de-coded part of the conversation we are liable to be lost (as we might be when listening to an imperfectly learned foreign language). The mathematical argument, however, need not be de-coded until the end, particularly if the proof has been formal or reasonably standard and we are really only interested in the result. This is the 'computer' aspect of work with symbolism. A mathematician can of course de-code at a moment's notice; he just does not do so to save time and energy. The child at school playing a game with symbols might not be able to de-code in the middle, but only at the end. If this is so, then we have merely programmed something into him which will be started by a certain type of cue and mechanically performed until the programme has been run. It is then necessary to teach such a child some special tricks to de-code at certain frequently occurring points; namely points that are normally covered by examinations.

These last two categories, symbolism and interpretation, are not really separate as far as the patterns of mathematical thinking are concerned: they are what is externally observable during certain parts of such thinking. The reason it may be necessary to adopt them as categories nevertheless is because they very often occur in isolation. They occur, that is, not as an accompaniment to thinking but as autonomous activity a person has been trained to perform in order to reach a certain external goal that has nothing to do with mathematical thinking (e.g. the passing of examinations). There are, too, other ways in which symbolic work might take place, even without the operation of an external goal. Just as the electronic engineer might get interested in the working

of his computer, so the mathematical 'engineer' (he would be horrified to have himself so described, because he would be a 'pure' mathematician and usually a formalist) might get intrigued with the mechanism of the symbolic-computer and begin to play with it, or even build new parts where he thinks they would fit nicely. He would call the latter extensions or generalizations. If he went on long enough without any regard to the mathematical entities that his new symbols symbolized, he could get completely out of touch with 'reality' and just become involved in an intricate game. Of course maybe ten years later somebody might find some very important use for his game, and so this kind of formal playing about should not perhaps be too much discouraged; but it is safe to say that in the case of children the games should be kept anchored to the ground until it is quite certain that the young mathematician is fully in control of his fancy games.

One example of this formal playing is the study of multi-valued logics defined by matrices, and those defined by axiom systems. It is fascinating to consider what could happen if we had three values in our logic instead of two, with certain given 3 by 3 truth tables: whether it is possible to invent an axiom system which generates those and just those statements out of the axioms that turn out to be tautologies in the 3-valued logic. Or, conversely, given a system of logical axioms, whether there is a finite logical matrix which will yield as tautologies those and just those statements-combinations that are deducible from the axioms by certain admitted rules. This is unashamedly 'playing with a new toy', and as such is a fair mathematical game.[1] Maybe by introducing children young to mathematical games it is possible for them to play at mathematics on this formal level. We must be careful, however, not to confuse this with sheer jiggery-pokery aimed at examinations or even just pleasing the teacher.

3.4. *Cognitive model for the higher-order processes*

When an isomorphism between two types of situation has been established, something which is common to both has been abstracted from them. A certain amount of noise has been eliminated. Some, however, will remain. Children learning with wooden material may believe that what they have learned is to do with pieces of wood, even

[1] Dienes, Z. P. 'On an implication function in multi-valued logic', *Journal of Symbolic Logic* (June, 1949).

though they will see that the shapes of the pieces are not relevant when, for example, volume has been the relevant concept. To eliminate further noise the field of operations may be extended to yet other types of situations. How abstract one regards an actual situation to be will depend on the amount of noise discarded; i.e. on whether and to what extent one considers the situation is an exemplar of a class of situations from which noise has to some extent been eliminated.

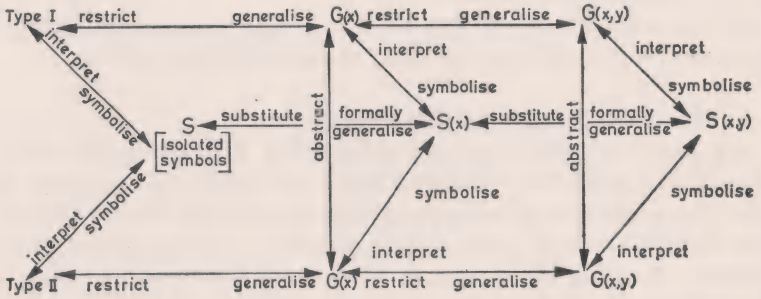
Generalization takes place when existing situations are readily extended to imaginary situations which have not served as exemplars but which are essentially of the same kind. If the generalization is explicitly made, that is, if there is an awareness that a certain rule-structure extends to a wider set of instances than was previously thought, then it can be symbolized in this general form. But the symbolism will not be fully operative unless actual situations that are symbolized by the symbolism can be given. There must be two-way traffic between a generalized situation and its symbolization. The reverse process of symbolization is what we have termed interpretation. The symbolism may or may not represent an abstraction. If the symbolism is arrived at as a way of communicating the common properties of different types of situations of the same structure, then the symbolism symbolizes an abstraction. In fact it may only symbolize a generalization along a very narrow front, or it may not even symbolize a generalization at all but particular situations not yet generalized. In this case the symbols are just used as a kind of shorthand for the subject's mother tongue for describing as yet unconnected experiences.

For generalization to be a functional part of a person's thinking the reverse process, particularization or restriction, should also be easily performed. The generalized structure should include in an accessible form the more particular structures from which the general structure was extended.

Generalization may take place formally. The class of numbers to which a certain structure is applicable may be formally extended by a kind of play with symbols. Formal substitution of elements of the previous, restricted, class will give rise to formal particularization. For this kind of generalization and particularization to be considered as mathematical thinking, it must be possible to interpret these in relevant concrete situations. When a generalization has lost sight of its content it becomes a useless and empty game, and will probably be

given up. It must be remembered that, in the higher reaches of mathematics, we often mean by content some previous structure, whose content is some still previous structure, and the final content of real experience behind it all may be so far away that we entirely ignore it. This rarified game is all very well for the expert mathematician who can at any moment retrace his steps to the reality entrance of the labyrinth of his own making, but we should do well to initiate children bit by bit into the mysteries of these games if we do not want them lost in the maze. Formal generalization is a kind of leap into the maze without the thread which will ensure that we can find the way out again.

These considerations lead to the following type of cognitive organization of mathematical learning as exemplifying the abstraction-generalization process:



On the diagram, abstraction is regarded as the potential activity of *being able to* travel from one element of a class to another. This ability becomes, with use, a mental set towards 'all those kinds of things' which will tend to be classed together. It is possible to generalize more than one variable at a time. This means that several of the generalization situations may be skipped. In the reverse direction it is likewise possible to particularize several variables at once. It goes without saying that there could be more than two types of situation. The 'cognitive map' above is still a relatively simple one. Logical analysis, for example, is entirely left out. When a generalization has been made, 'logical' paths lead to what we might call analytical insights (as we saw in the 'play diagram' in the first section). The 'cognitive map' can be considered as a cell forming part of an organic learning process in which there are cells of different structures, and each structure as adapted to

the particular kind of learning it serves to accomplish. Perhaps such an organic model might serve as a better model for developing learning theory than the current Stimulus-Response models, even when based on elaborate stochastic theory.

4. *Practical functioning of mathematical thinking*

In the previous section at least some of the salient features of mathematical thinking were defined or described. The problem that remains is how these various processes fit together to form the live thread of mathematical thinking. The various processes described cannot be considered in isolation; there are many other variables, psychological and mathematical, that will eventually have to be drawn into the picture before giving a coherent explanation of what really goes on when we think mathematically.

Let us begin by considering the abstraction process and trying to examine this in relation to the other mathematical processes.

4.1. *Abstraction through multiple embodiments*

We assumed throughout our experiments that abstraction would arise from a multiple embodiment of the concepts to be abstracted. By this I mean that situations physically equivalent to the concept-structure to be learned would, if handled according to specific instructions leading towards the structure, result in abstracting the common structure from all the physical situations. We thought that when this had been accomplished, symbolism could be introduced to describe the structure just abstracted. But as we observed the children going through such 'abstraction exercises', it soon became clear that the picture was far more complex than we had assumed.

At first there was the theoretical difficulty of finding the criterion for determining whether a concept had been learned as an abstract structure or not. According to our first 'simple-minded' theory, abstraction was tested by children's ability to transfer a mathematical structure embodied in one situation to a different embodiment. As we shall see, there are certain difficulties about this procedure, and, even theoretically, in order to make the procedure valid some method of comparing embodiments would have to be devised. Clearly some embodiments are closer to each other than others: in one case, for example, only a small number of the attributes irrelevant to the structure may have been

varied, and in another a large number of these attributes might have been varied. It seems that it would be possible to make an embodiment so 'noisy' that extracting the looked-for mathematical structure might be well-nigh impossible. It needed a Newton to discover that there was a common structure underlying the phenomenon of falling bodies and the orbit of the Moon around the Earth—there was a great deal of noise to break through, and only a genius could do this. One day we might be able to measure the extent to which a subject conceives a structure as abstract, by measuring the amount of noise he is able to break through while still seeing the structure. In the experimental groups reported here, not more than five different embodiments were used for any given mathematical structure and although these fell into recognizable pairs no attempt was made to construct an 'embodiment space' with a metric therein, though this would be necessary before we could speak about the 'distance' between embodiments.

Apart from these theoretical difficulties, it was not long before we encountered practical difficulties as a result of the 'simple-minded' theory once the sessions with the experimental groups were under way. As I have said, 'tests' were given to the children in Group A to ascertain whether they could transfer from one embodiment to another; that is, whether they had apparently formed any kind of isomorphism between the two physical situations where transfer was being tested. Such isomorphism we thought equivalent to a certain degree of abstraction of the mathematical structure exhibited in the physical situations. Below are examples of such tests.

Isomorphism Probe after Unit 1

TO OBSERVER:

Lay out six different arrangements of three groups of four squares. Ask child how he can lay out pegs on pegboard so that it will be in some way like all the arrangements. Then ask child in what way his construction is like all of the six displays of three groups of four.

Unit 2 Card P.14

Put 6 rings on hook 6 on one side and 4 rings on hook 9 on the other side. Put two piles of 36 squares each on either side of the balance. Can you do something with these squares that would remind you of what is on the balance? Have a try.

Unit 2 Card P.15

Make two rectangles, each with 24 squares in it, one 6×4 on the left, the other 3×8 on the right. Try to do something with rings on the balance that would remind you of what there is in the rectangles you have made.

Unit 2 Card P.16

Take 21 beans and 7 cups, and put all your beans in the cups. Put the same number of beans in each cup. Put these cups on the left of the balance. Now take another 21 beans and 3 cups, and put these beans in the 3 cups, but put the same number of beans in each cup. Put these cups on the right of the balance.

Now try to do something with the rings on the balance which would remind you of the two arrangements of beans in the cups.

One practical difficulty was that the children interpreted the question 'In what way are they alike?' in a composite way. In the isomorphism probe three of the four children in Group A, when asked to transfer from 3 groups of 4 objects of various kinds on to the pegboard using pegs to put in the holes, made patterns in the pegboard where they represented all the *shapes*. In these children's eyes something which is like situation A, also like situation B, also like situation C and so on, must have all the properties of all the situations, and not just one property which is shared by situations A, B and C. This sense of *pictorial communality* of all the situations came more naturally to the children than the extraction of something *common to all* the situations. In this case the concrete or perceptual properties of the actual collections were a hindrance instead of a help in finding what was common mathematically to all the collections.

By contrast, all further transfer tests were done successfully by all members of the group. It must be remembered that the isomorphism probe was the very first transfer test, that the children had only handled squares and cubes, and inter-embodiment communication was an entirely novel idea to them. The following are the protocols of one of the children for both the isomorphism probe and the transfer tests two days later.

Example 13 Group A Pictorial communality

(after reading the Isomorphism card)

S. 'This is going to be hard.'

D. 'I want you to build something like every one of these in some way.'

S. 'Like one or like all?'

D. 'Like every one in some way.' (She puts in four pegs in the following pattern)

X
XXX

S. 'Now I've got the triangle system.' (Then she adds more to make the following pattern)

XXX
XXXX

S. 'Do you have to stick them in or can you lie them down?'

D. 'Anything. How is what you've done like this?' (He indicates 4 groups of 3 matchboxes)

S. 'It's not like *that* one.'

D. 'But it is supposed to be.'

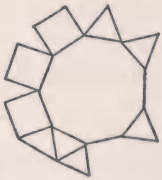
S. 'Well, you can make the squares out of triangles.'

D. 'Come and look at it from over here.' (S gets up and walks up to another side of the table)

S. 'They're all designs. Hey, they all have fours in them.' (She counts several groups of fours, three or four different times) 'I can make lots of four designs.' (She goes to build. She builds the following design, with the pegs lying down)

S. 'Those' (indicating the boxes) 'are squares, aren't they? Four-sided figures?'

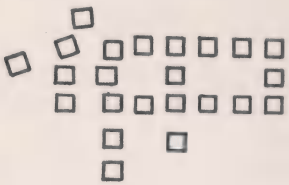
D. 'Yes.' (She continues and builds up the following pattern and has the pegs lying down)



S. 'There! It's got lots of triangles and some have almost the exact same design, and it's got the best circle I can make.' (Then she takes blocks to build)

D. 'What are you trying to do?'

S. 'To make a design like all of those.' (She makes an elaborate design)



(She built this with great concentration—a circle in the centre, a rectangle on one end, and the other blocks carefully placed in relation to their immediate neighbours rather than to the whole.

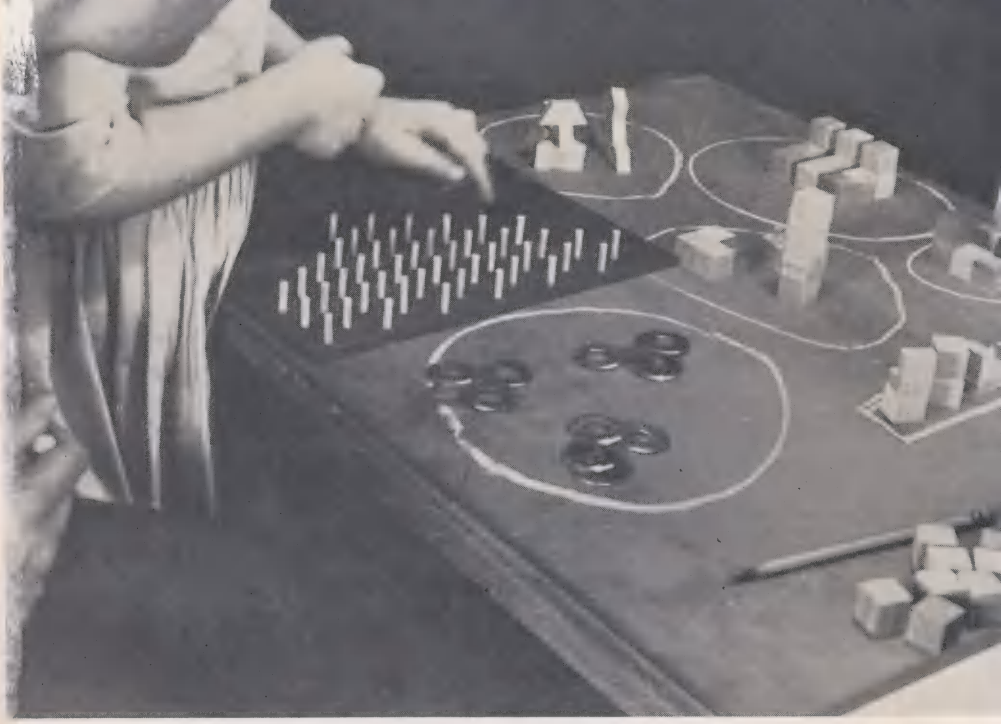
S. 'Does it all have to be hitched?'

O. 'Come over here and look again. What had you found the last time?'

S. 'Fours. All I'll do is fours, all different designs of fours.' (She starts taking the small wooden triangles and tries to make squares from them. At this point D says that that's fine and she can go on to another card)

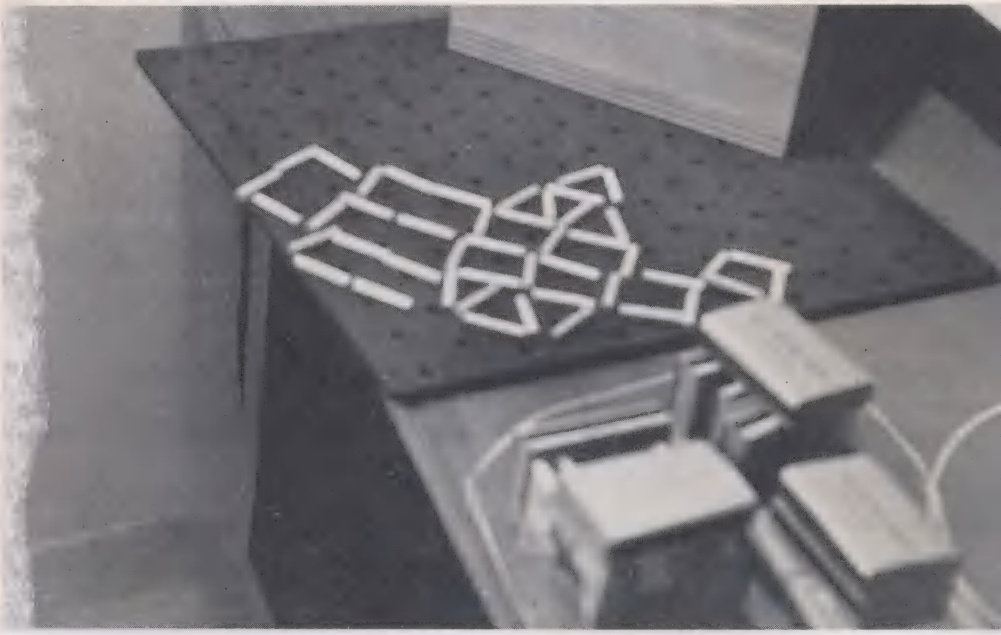
Example 14 Group A Isomorphism

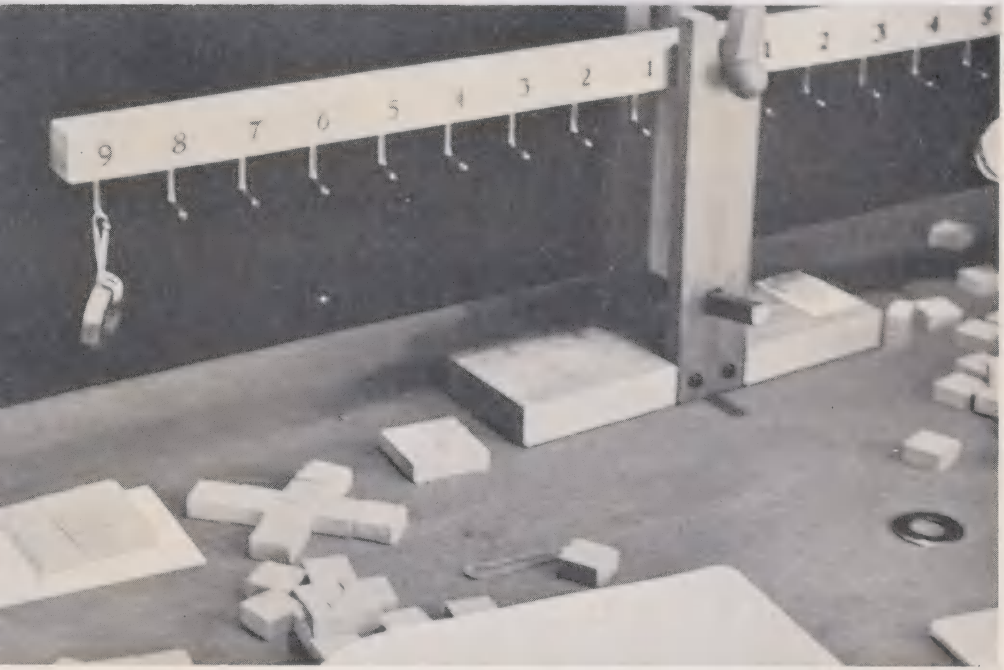
(S goes right to work on Card 14 and put the rings on the hooks of the balance. Then she takes the first group of 36 squares and tries to



Initial attempt at isomorphism (p. 69)

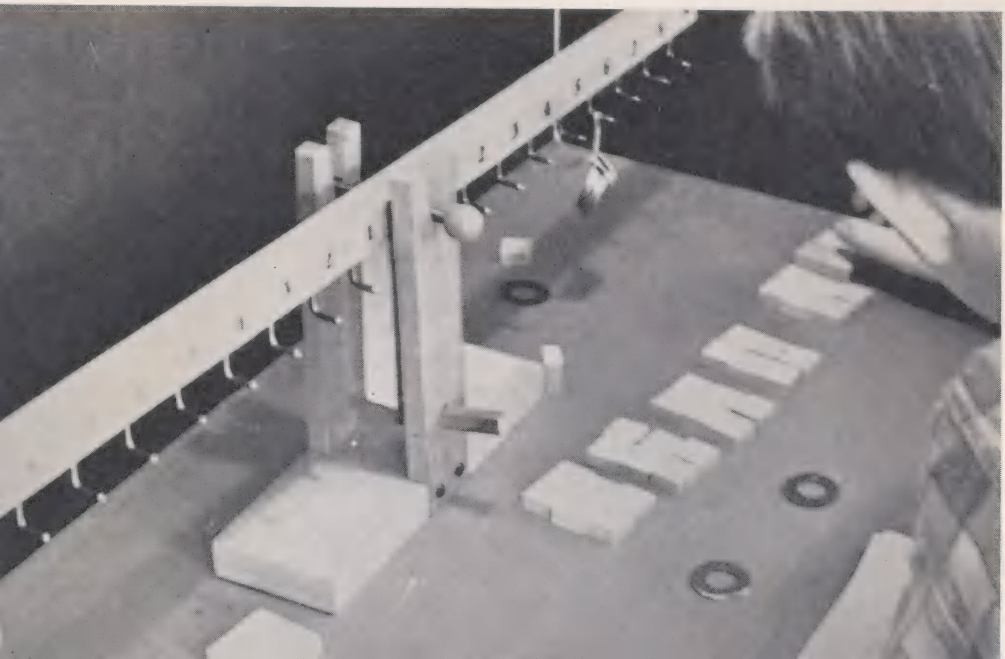
Isomorphism probe—pictorial communality. Notice squares, rectangles, triangle and even an attempt at a circle, to incorporate the rings (pp. 71–2)





Left side of balance. Notice semi-playful way of indicating multiplication by making symbol out of wooden blocks (Card P.14. Unit 2—not described in text)

Right side of balance. Here 6×6 is 'embodied' instead of symbolized by blocks (Card P.14. Unit 2—not described in text)



represent the 4 rings on hook 9. She builds a wall 4 blocks long and 3 layers high. Then, at right angles to this, extending behind it, she puts another 3 blocks and builds this up 3 blocks high also so that she has 2 walls of 4×3 and at right angles to one another. This construction falls down as she tries to fill it in)

S. 'I was trying to do 4 long and 4 wide and everything because there's 4 on the 9. But now I'm going to do something else.' (Flat on the table she builds a rectangle; she builds first a side 9 blocks long and then a side 4 blocks wide and fills in the whole rectangle) 'See, there's 4 wide and 9 long.'

O. 'Is it like the 4 on 9?'

S. 'Yeah, I think so. Now I've got another thing planned for 6×6.'

O. 'What?'

S. 'A wall.' (Then she puts 3 blocks, one beside another, 6 layers high and then starts again to build a wall at right angles 4 blocks deep)

S. 'They both equal 36.'

O. 'How do you know?'

S. 'I'm positive, because 6×6 is 36, and 9×4. Besides, I found out; I used 36 squares there and it worked.' (She continues to build up the wall in depth behind the 3×6 wall that she has built) 'It doesn't work, I know.' (The front falls down by error) 'I'm going to make a wall. No, I'm not.' (Nevertheless she builds a square 6×6. There is one block left over) 'Hey.'

D. 'What's the trouble?'

S. 'I don't know.' (She counts the blocks one by one up to 36 and exclaims) 'Dr D gave me 37.'

O. 'What were you trying to do before?' (referring to the wall which she had been building)

S. '6 high and 6 back, but there was too little' (too few blocks)

O. 'How is it the same as what's on the balance?'

S. 'There's 4×9, and I've got 4 washers on hook 9. There's 6×6 and I've got 6 washers on hook 6.'

(S was presented with Card 15. She built the required rectangles. Then above the 3×8 rectangle she put 8 rings on hook 3. Then she took 4 more rings)

S. 'What did I do? I put 8, then 6.' (She adds 2 to the 4 rings she had taken and puts these 6 rings on hook 4) 'Hey, these balance.'

O. 'Were you going to do something with 4 rings for a while?'

S. 'No, there's 8 on the top and 3 on the side' (in her rectangle) 'so there's 8 rings on 3. So 6 on the top means 6 rings on 4.'

O. 'Can you do it with less washers?'

S. 'Hmmmmm. Yes.' (She puts 4 on hook 6 and 3 on hook 8) 'It balances!'

O. 'How many blocks do you have?'

S. 'Hmmmmm. 6×4, 24. That's cinchy. 24 on each side.'

O. 'Does that surprise you?'

S. 'No, you could have any number. You could have 100 on each side.' (Then she takes 2 groups of 12 rings in each group) 'I wonder how much 24 washers would weigh?' (She attaches each group of 12 rings to hook 9 on either side. Then from the right-hand side she moves 12 from hook 9 to hook 8. Then she plays with 5 other washers, trying to balance them. Finally she puts 3 on hook 2 on the left-hand side and 2 on hook 9 on the right-hand side. Altogether she has 12 on hook 9 and 3 on hook 2 on the left, 12 on hook 8 plus 2 on hook 9 on the right, and it balances) 'What's 12×9?'

O. '108.'

S. '108 minus 9, that's 99, then 108 and 6 is 114, and 99 and 18 is 117.'

Bruner. 'Do you know what 12×8 is?'

S. 'Isn't that 9 less than 12×9?'

B. 'No.' (He explains what the difference is, and says that 12×8 is 96)

S. '96 and 18, 114. 108 and 6 is 114!!! Oh, I've got a wonderful experiment and it worked.' (S puts her thumb in her mouth, very happily. She looks up at Dr B and says, 'I still suck my thumb sometimes. I love my thumb.' (Then she put the rings on her thumb) 'The rings are sucking my thumb now.'

(After she has recovered from the excitement of this discovery, she goes on to Card 16.)

S. 'There are 7 cups, so 3 beans in each cup.' (She puts 3 beans in each cup.) 'I'm going to plant beans.' (Then she starts recounting a long

involved story about what will happen when she has planted the beans in relation to a natural science class that she has had. She ends by saying) 'And then it will grow into a long story of science and math.'

(She goes on with Card 16 to read, 'Put these cups on the left of the balance', so she takes the cups and literally puts them on the balance beam, balancing them there. Then she puts all the cups one beside another. She anticipates the rest of the instructions and takes 7 cups, guessing what to do without reading, once again putting 3 beans in each cup)

O. 'But 3 cups.'

S. 'Oh, then there will be 7 in each.' (She empties every second cup into the cup beside it and then from the last cup takes the 3 beans and puts 1 in each of the 3 cups to get 7 beans in each cup. Then she takes these cups, puts one inside the other, tries to balance on the balance beam the 7 cups with 3 beans, then 3 cups with 7 beans)

O. 'Why doesn't it balance?'

S. 'Because the cups weigh more than the beans.' (She continues again to try the balance beam, hanging the cups on to the hooks of the balance. Then after a while she goes back to the card. She completes the task, putting 3 rings on hook 7, for the 7 beans in each of 3 cups, and 7 rings on hook 3, for the 3 beans in each of 7 cups)

The process of noise-reduction through inter-embodiment transfer is not easy to initiate, but once initiated it does seem to lead to successful abstraction. It was upon this principle that most of the work in the Leicestershire Mathematics Project was based. Achievement of the abstraction was tested in this Project by providing verbal problem situations whose structure was the same as that of the mathematics which had been abstracted.

In the next section we shall give an example of a perceptual block which arose directly out of the embodiment experience to show that the road to abstraction is liable to be beset with difficulties.

4.2. *Embodiments interfering with abstraction*

If abstraction is cutting out irrelevant noise as well as realizing a common structure in many different situations, then the provision of

artificial noise in the form of multiple embodiments may impede as well as assist in the abstraction process. The subject may pick up the noisy aspect of the embodiment as essential, and so when confronted with another embodiment may search for this irrelevant noise in the other embodiment. If it is not obviously there he may even try to put it there artificially. Composing a total picture from the pictorial communality of all the embodiments as we saw happening with the isomorphism probe is an extreme example of this. *All* the noise was regarded as relevant and so had to be incorporated in the final picture. Part of the difficulty seems to be that what is to be regarded as noise is really quite arbitrary; it just depends on the particular mathematical structure we are trying to induce our subjects to learn. It seems *a priori* probable that the more embodiments a subject is presented with, the less likely he is to pick on the irrelevancies, particularly if the embodiments are so chosen that the same irrelevancy does not occur too often—ideally only once. In the example that follows the subject had been working possibly too long with the same embodiment, and became over-attached to its perceptual qualities. The perceptual qualities interfered with the abstraction process and the subject was unable to transfer the *mathematics* of the first embodiment to the second.

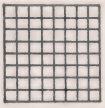
It will be worth quoting the entire protocol as it is an extremely interesting example of a breakdown. One can see that the subject had little or no trouble generalizing the situation, and very soon realized that one large square, four strip lengths and four extra small squares always made another square two small squares longer than the original large square. She could predict accurately on this point and appeared to feel quite at home. Yet as soon as an attempt was made to transfer the situation from the strips and squares to the mathematically equivalent one of cups and beans, the perceptual quality of the square always got in her way. Even the introduction of the garage cover story by Dr B did not appear to help. Clearly a square number was not mathematical currency; the idea of as many groups as there are objects in each group (i.e. the mathematical square) had simply not been formed, and this is shown by her reluctance to accept $L \times L$ as expressing the number of small squares in a square L small squares long. Expressed in terms of cycles, one might hazard the suggestion that since the cycle leading to the abstraction of the mathematical square had not been completed the subject was resisting attempts to turn this unfinished concept into a

grammatical subject which was to be bandied about, talking about it, adding things to it, making other specimens of it and so on. On the other hand, within the narrow perspective of manipulating squares she was able to see a general pattern and predict accurately. Here we see on just how narrow a front learning may proceed if we do not try to broaden it by making sure that the situations handled are appreciated as exemplars of a whole class of situations.

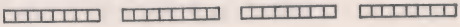
Example 15 Group A Perceptual block

I

Make a square out of little squares. It might look like this:



Then make four 'strips' out of squares, each 'strip' just as long as one row in your square.



Then take 4 more squares.

Try to put all these together to make a larger square.
Your figures need not be the same as those on this card, they can be smaller or larger.

How much longer is your larger square than the one with which you began?

II

Let us use the code letter S to stand for the number of squares in a strip. How would you write down the total number of squares in 4 strips?

How would you write down the total number of squares in a square in which every row had S squares in it?

How would you write down how many little squares there were altogether in one large square in which every row had S squares in it, another 4 strips of S squares each, and 4 more little squares?

How would you write down how many little squares there were in one row of the big square you could make with all this?

(S began the new series which was an attempt to help her see the matrix relationship of the square. She built skilfully and accurately, making the square as indicated in the directions)

D. 'How much longer is your larger square than the one you began with?'

S. 'I'll have to figure.'

D. 'You don't need to figure.' (S, however, counted the total number of squares. She did not see the relationship. S then built a 3×3 square, added the strips and small squares)

S. 'The strips are 2 longer.' (S then constructed a 5×5 square) 'You always have 4 strips and 4 extra squares.' (She constructs skilfully and systematically, moving 2 strips in the length and 2 for the width, and fills in 1 square and then fits in the other 2 strips and then 3 squares. She does see that the square is now 2 squares longer. She then works out a symbol system with D)

D. 'If there are 5 squares in a strip, how many in the big square?'

S. '25.'

D. 'If 4?'

S. '16.'

D. 'How did you get it from the length of the strip?'

S. 'I added them.'

D. 'How many in the bigger square?'

S. 'With the small squares, you get the big square.'

D. 'How?'

S. '5×5.'

D. 'From the length of the strip calculate the number of squares in the big square. For example, if you have 10?'

S. 'Then you would have 100.'

D. 'Use 10×10 for the square, 4 more strips, 10 squares long, and 4 little squares.'

S. 'Then it would be 100 and something.'

D. 'How long would the square be? How long, not how many squares.'

S. '12.'

D. 'How many squares then altogether?'

S. '12 something, 12×12.'

D. 'That's 144. You haven't learned that yet in arithmetic.' (Then D and S went on to symbolizing the relationship) 'The big square is 10×10.' (Then S worked out $10 \times 10 + 4 \times L + 4 \times S = 144$ or 12×12) 'Let's try another.'

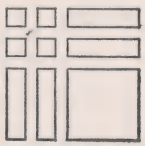
S. '$9 \times 9 + 4 \times L + 4 \times S = \ldots$'

D. 'How much longer would the square be?'

S. 'It would be 11×11.'

D. 'And that would be 121. Each time we have to alter a number and the total. Let's take another number.' (She wrote $7 \times 7 + 4 \times L + 4 \times S = 81$ or 9×9. D then pointed out that each time she had to alter the total and the number of squares in the original square) 'Let's write it so that you don't have to alter it.'

S. 'Is it possible? Are you sure? Let's choose 13.' (She saw immediately that the resulting square would be 15×15 and she wrote $13 \times 13 + 4 \times L + 4 \times S = 15 \times 15$. D then separated the arrangement which was on the desk)



(D pushed on to the $L \times L$ relationship. S appeared to have insight that the small square could be represented by $L \times L$)

D. 'There are LL's in that square.'

S. 'Did the third grade do this last year? Oh well, all right . . . $L \times L + 4 + 4 \times L$.'

D. 'How much is it equal to?'

S. '25.'

D. 'How much longer? How much more than L is the larger square?'

S. '2 longer.'

D. '2 more than L long each way. How do you write it when something is 2 more?'

S. 'Add 2. Then it would be $L+2\times L+2$.' (S worked out the relationship, first by reconstructing the 5×5 basic square, using the 4 strip lengths and the 4 squares to arrive at a 7×7. Then she wrote $L\times L+4+4L$. S walked away to play with J. This time it was a game of rattlesnake. D took advantage of this, asking her to work out what she had been doing with cups and squares. S put 4 squares in each cup)

D. 'Show me an $L\times L$ with cups and squares.'

S. 'I don't know how to.'

D. 'An $L\times L$?'

S. 'All right. I'll do 4×4.' (Then she laid out the following arrangement)

(The geometry of the square is interfering with the transfer exercise. S then made a similar arrangement)

Then the session ended.

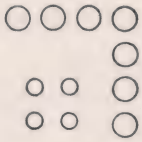
(Next day S began the session by playing the game of rattlesnake she had started the day before)

O. 'Do you remember what you were doing yesterday?'

(S did not respond to this cue and continued to play until D came by and suggested that she made a square and then made the same thing with cups and the little squares. S then built a 4×4 with the small squares. However, she continued the play activity by embroidering this square with a design, describing it as the King's crown. D then separated the 4×4 square into a 2×2 square—4 strip lengths and the 4 small squares—and then continued)

D. 'How would you show someone with the cups that there are 4 squares in the big square, 4 strip lengths in the 4 squares and 4

small squares?' (S laid out the following arrangement with the cups and squares.)



(S commented that the cups running along the top of this arrangement were really the strips. D then suggested that she try a 5×5 big square—4 strip lengths with 5 small squares in each and 4 small squares—to see if the geometry could break down. S laid out the square arrangement rapidly and skilfully. She is extremely adept at constructing but when she transfers to the cups and beans she continues to lay out a geometrical pattern with the cups. She does not see that the 5×5 squares should have been represented by placing 5 squares in each of the 5 cups. She would put 4 in each cup)

D. 'Are you sure that you know what you are doing?'

(Then he wrote)

5×5	square
4 strips	(five squares in each)
4 little squares	

(S then wanted more cups. D reviewed with her the number of squares. S continued to confuse the cups and the squares in attempting to embody the relationship. S completed the session by constructing rapidly and skilfully a series of squares 7×7, 8×8 and a 9×9. As yet, she has not achieved insight into the mathematical concept of a square)

(The next day, S was given a new problem)

Special Card

You remember last time you made a square out of one 5×5 square, 4 strips (5 squares each) and 4 extra squares, and it made a 7×7 square? Then you tried to put these squares into cups to show separately

5×5 squares (First pile)	4 strips of 5 squares each (Second pile)	4 squares (Third pile)

Try to do this now (there was no time left yesterday). Then do the same with

6×6 squares 4 strips of 6 squares each 4 squares

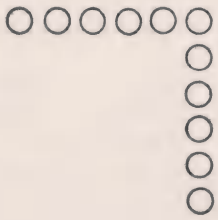
again putting them in 3 separate piles.

Now you can arrange the first of these to show 7×7 squares (in cups), then the second one too, showing the larger square you can make but again putting your squares in the cups.

How many more cups do you finish up with than the number of cups in the first pile?

(S built a 6×6 square and a 5×5 square following the directions on the card)

S. 'I know that it's going to be 8×8.' (This was in relation to the 6×6 square, 4 strip lengths and 4 squares. Then she set up the cup arrangement as follows)



(S pointed to the row and said that these count for strips. You pretend that these are strips. She constructed a 7×7 square from the 5×5 squares, the 4 strips of 5 squares each and the 4 squares. She then broke the 5 square down into the 5×5. When she was asked how many squares in each side of the square, she counted 9. She had difficulty in seeing the row-column relationship with their nine elements. Then B began to work with her, telling her that he could show her that four hands would have only nineteen fingers. She saw the ruse immediately, saying that he didn't count one finger when he should have. B sketched out a 5×5 square saying that each column was a garage and that there were five cars in each garage. He then asked her to show him what the five garages with the five cars in each garage would look like with cups

and squares. S did not see the point. Then B asked her to draw something like the five garages and the five cars. She sketched out a Christmas tree with five branches to which she appended circles)

It may be worth noting that this subject proceeded to make a successful generalization of the $X^2+2X+1=(X+1)^2$ situation on the balance beam, although not with the generalized symbolism. There was no evidence that she saw any connexion between the balance beam exercise and other exercises on the binomial situation.

The sessions ended a day or so after the time to which these protocols refer and it was not possible to pursue the problem of whether the subject would eventually form a transferable, abstract structure from her experiences.

Example 16 Group A $(x+1)^2$ on balance

Special Card
Balance Beam

Choose a hook on the balance, but not number 1 or number 2. Put the same number of rings on your chosen hook as the number of your hook. Then put the same number of rings on hook 2. Then put 1 ring on hook 1. Balance all this on the other side on just one hook, making sure that the number of rings you use is the same as the number of the hook you use on this side.

Do this several times. If your chosen hook had the number B, what was your hook number on the other side? What was your ring number?

Is your hook number on the other side always the same amount more than your chosen hook number?

(S began work immediately on the problem. She liked the balance beam so there was no difficulty in getting her to focus on the task)

D. 'What hook? Do you want to choose?'

S. '9.' (And she proceeded to put 9 rings on hook 9, 9 rings on hook 2 and 1 ring on hook 1)

S. 'I do know how to balance everything. I can do it by 50 on 2, 20 on 5, 10 on 10 and 100 on 1.' (She then realized that she had to use the

same number hooks as the number of rings. She then balanced 10 on 10)

D. 'If you put 7 on 7, 7 on 2 and 1 on 1, how would you balance it?'

S (immediately). '8 on 8.'

D. 'You can do this in your head. You can work it out without adding what is on the right side and then balancing on the left. You can figure it out immediately. Look at the way it's working out.'

S. 'I like to do it the harder way. I don't like to guess.' (Then she set up 8 on 8, 8 on 2, 1 on 1) 'It's got to be. If it's an even, you add one, and then you get an odd number.' (D recapped by writing $8\times8+8\times2+1\times1=9\times9$)

D. 'Let's try it with 10.'

S. 'Then it would be 121 and that would be 11 on 11. You go up 2 each time, 8 times 8, 8 times 2, 8 times 1, that's 81 and 19 and that gives 100.'

D. 'Can you guess how it is always going to go?' (S then took time out for a drink of water)

Time: 9.11

(S returned and played around until 9.15. S then set up $7\times7+7\times2+1\times1=8\times8$ on the beam)

S. 'If you add 7 and 8 you get 15 and then add 15 to 49 and get 64 and then you balance with 8 on 8.' (O then wrote

$$8\times8+8\times2+1\times1=9\times9$$
$$7\times7+7\times2+1\times1=8\times8$$
$$6\times6+6\times2+1\times1=7\times7$$
$$5\times5+5\times2+1\times1=6\times6$$
$$4\times4+4\times2+4\times1=5\times5$$

in an effort to help achieve the relationship involved)

O. 'If the hook that you chose had the number 8, what would be the hook number on the other side?'

S. 'It would be one more.'

Session ended.

4.3. *The forming of isomorphisms*

A central problem in successfully accomplishing the abstraction process is inducing subjects to form isomorphisms, this being the positive facet of the process. The negative facet is discarding detail irrelevant to the structure being learned. Some subjects appear to think to such an extent in terms of classes, i.e. in 'those *kinds* of things', that when another exemplar of 'those kinds of things' turns up they recognize the similarity and it appears that they form the isomorphism in an instant. In fact they probably do not form it, as it is implicitly formed already. Others do not have such a classified view of the world and they will need some help. It is for such people that artificial exercises in forming isomorphisms could act as teaching devices to help them recognize similarities when they see them. To what extent abstracting ability may be improved by means of such devices is an open question.

Another problem connected with the abstraction process is one of development. It is fairly certain that on the whole an older person can cut through more noise than a younger person, but it would be interesting to know more about the detail of this kind of development. Group B which consisted of seven-year-olds definitely found the isomorphism exercises very uphill work, and it is possible that we aimed at a breadth of abstraction that was too difficult for such children. The following account will make clearer the kind of difficulties they were grappling with.

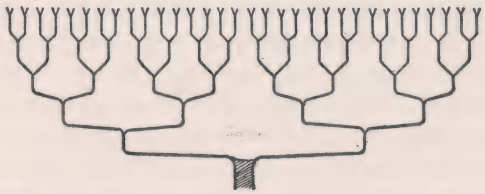
Example 17 Group B Tree and blocks games

We felt that the development of the isomorphism between the two embodiments of the multibase arithmetic ideas[1] did not proceed rapidly enough. It could be conjectured that the difficulty was a developmental one, in which case no amount of further 'teaching' was likely to have much effect. On the other hand it was possible that children could be trained to look for and to establish isomorphisms. We decided to aim at the concept of inequality, structured to the place-value system. The method was to play the following games on the hypothesis that the children would be more likely to form a simpler isomorphism that did not involve a lot of counting and comparing

[1] See Appendix I, p. 189.

tables of numbers, etc., and by studying many successive isomorphic structures would form a set for isomorphisms.

(A) *The tree games.* Structures such as the one below were drawn on hard paper. Cubes were turned into dice by writing appropriate numbers on the faces, such as 0 and 1 for the tree with double branching; 0, 1 and 2 for the tree with treble branching; 0, 1, 2, 3 for the tree with quadruple branching and so on. You moved up the tree according to what you threw on your dice, the 'values' of the branches always increasing from left to right. The person who finished furthest to the



right at the top of the tree won the game. Records were kept of all the throws so that past games could be reconstructed and the winner determined.

(B) *The blocks games.* In this the MAB blocks were used, and the game was again played with the appropriate dice. The first throw was for the third power, the second throw for the second power, the third throw for the first power (or the base), and the last throw for the units. Each child took as many of these pieces as the number that turned up on the dice that he had thrown. Since the tree game went up to higher powers than the third, the blocks game in base 2 was started at higher powers (fifth or sixth).

The blocks games could of course equally be played with either embodiment of the multibase arithmetic.

Six children were taken at a time, divided into two groups of three. In each group one child would record the throws on previously prepared record sheets while the other two played the game with the dice and moved up the tree, marking their positions during the course of the game by using representative pieces. Having played six games or more the children were asked to say again, from the records of the games, who won which game. Nobody could remember all the games, or in what

order they came, so the task had to be done from the records. There was a great deal of disagreement about who won the games, some children insisting that if you had a greater number of large numbers you did better than if you had a few. The arguments were of course settled by replaying the games from the records. Fortunately one of the games was actually a tie, and so the children had an immediate instance of an equality. On the other hand they would still sometimes judge a game a tie if both children had the same throws even in a different order. The only circumstance that readily convinced them was that they knew that there had only been one tie. They then verified that indeed you could have the same throws, and yet one would do better than the other. This forced open the situation about the relevance of the order in which the numbers occurred.

Some imaginary games were then put on the board, and the children had to guess the winner. The boys here were much quicker than the girls; in fact they would point out the winning throw during the games and it was on occasion difficult to persuade them to continue as they said the question had already been settled because you could not jump from one branch to another. This was readily transferred to the 'digit' notation, and for example a comparison like

2303 with 2331

would be immediate, so they realized that the third throw had won the game. The girls had to go back to the tree much more before they could be sure, and even when they did not look at the tree, you could see that they were constructing a tree in their heads and trying to imagine where they would go on it. The boys had picked out the relevant piece of information, namely that the first throw which was different was all one needed to look at, and so they could dispense with the use of overcumbersome imagery.

The blocks game was played to the bases 3 and 4 with one group quite extensively, each child building his own towers quite systematically out of his 'winnings'. Records were kept of the games as before. When they reached the point where they had to 'guess' who won, they were right back at the stage of thinking that a greater number of larger numbers would win the game. They were so certain of these false conclusions that it was difficult to get them to verify them by building up the corresponding towers. They were genuinely astonished when they

had discovered their mistake. There was at this stage, however, no realization that the rules of the blocks game were precisely the same as the rules of the tree game. The isomorphism had simply not been formed. The games were quite different games for the children, even for the boys.

While one group was grappling with the blocks game, the other group was working on the following type of problem with the tree game:

One child gets 2 1 0 0. The other child he is playing with just gets one less point. What were his throws?

The problem was alternately given as one less or one more, and the children were encouraged to give each other problems and if possible to find how the thing worked so that they could do it without looking at the tree. Again the boys were quicker here and soon went on to tackle the problem of 2 more points, while the girls seemed to need the perceptual peg of the tree to keep in mind what they were really doing. One girl was particularly taken with this problem and at the end of the session, though repeatedly told that it was time to stop, went from the tree to the blackboard and back again a great number of times, looking very worried but determined to get to the bottom of the affair. She was clearly challenged by the isomorphism problem of game versus notation which she regarded as exciting and yet was unable at that time to solve.

This kind of practice was intensified for two more sessions by playing the blocks game along with the tree game, drawing on the blackboard, the children giving each other hypothetical games, determining the winner and so on. The point about the winning throw was emphasized, and this was readily seen by the boys but not so easily by the girls. Finally we also introduced a 2-game with the blocks, going up to 2^6 and so necessitating seven throws each per game. The inevitability of a win was seen much sooner on the tree game than on the blocks game; to sharpen the problem, a game like this:

I 0 I 0 0 I I
I 0 I I 0 0 I

was altered to:

I 0 I I 0 0 0
I 0 I 0 I I I

and the children were asked whether the change affected the winning of

the game. At last nearly all the children realized that a particular game would be won by the same person whether it was the blocks game or the tree game.

To round off the learning, the following paper and pencil exercises were given out:

Put a tick against the winner in each case

2-game:

Blocks	Trees	Bl.	Tr.	
I I I 0 0 0 I	0 0 I I 0 I 0			
I I 0 I I I 0	I 0 0 0 0 I I			etc. . . .

3-game:

Bl.	Tr.	Bl.	Tr.	Bl.	Tr.
I 2 0 I	2 0 0 0	2 0 0 I			
I I 2 2	I I I I	I 2 2 I			

and so on in a similar way for the 4-game and the 5-game.

Later the 10-game was played, but only with three throws as it was difficult enough to make the 10-game with one thousand branches at the top! The children were asked where the person who had, for example, 458 points on the tree game finished; then they had to say what pieces a person had in the 10-blocks game if his first throw was 4, the next 5 and the next 8, and eventually find out how many units he had altogether. The purpose of the exercise was to see whether the place-value notion was learned as an abstraction; the different bases had been used to make sure that they learned it as a generality. Not all the children reached this insight by the end of the sessions.

It may be that younger children generalize over a narrower front than older children, which is another way of saying that the breadth of abstraction of which a person is capable may increase with chronological age. There are a great many examples in the protocols where successful abstraction took place in the sense of widening the front of a generalization. The example already quoted of the successful transfer tests is one. Another clear example occurred in Group D when, after playing the magic game with the dance-floor in which there was a boy's magic and a girl's magic as well as a man's magic and a woman's magic, one of the children said, 'The boy's move is like the "nothing"

move and there are only three real moves. It's like two moves equal a third move.' This child was identifying the magic game with the second four-move game[1] from the structure of the moves, and so had cut through the noise of the magic and of the concrete operations of turning things or himself round to the underlying mathematical structure, which was in this case the complex algebra, though he did not know it.

Another example from Group D was a child who identified the particular transformation game being played with the magic game. This transformation game (referred to in the description of the games as complex algebra transformation game—Appendix I) was identified with the magic game. This is cutting through the noise in the opposite direction, making the transfer from the transformation to the algebra. It is probably in the same order of difficulty as the abstraction example quoted in section 3 on the two different interpretations of the complex algebra; in fact it is very nearly the same thing.

When abstraction and generalization take place fairly close together in time, there is often great excitement. Finding a general rule extending beyond the initial instances (generalization) and realizing that this rule applies in a host of different situations (abstraction) must indeed be a very exciting experience for children; it is probably the motivating power of this sort of excitement that keeps the professional mathematician at his work. An example of this is quoted below from the Group C protocols, where we introduced geometrical progressions.

In this example the children achieved a generalization; namely the general method of summing a geometrical progression. They then saw that this generalization applied in a wider situation than the one in which it was generalized. They realized the isomorphism between the two situations in a flash, as they probably conceived the generalization as an exemplar of a wider class, i.e. in the abstract, in the first place.

Example 18 Group C Geometrical progressions

The series to the base 3 was produced and the following instructions given: 'Start with the larger pieces and take two pieces at a time; try to

[1] See Appendix I for description of these and other games.

build a tower and fill up any "holes" in the tower as much as you can.' Very soon the subjects had made a representation of 3^9-1, starting with two 3^8 pieces, then two 3^7 pieces and so on. This resulted in almost a cube representing 3^9 but there was 'a little bite taken out of it', the size of one unit cube, as one of the children aptly remarked. They then repeated the exercise with one piece each in base 2, 3 pieces each in base 4, and 4 pieces each in base 5. There was great excitement each time at the 'bite' that kept on appearing. This experience was eventually symbolized into

$$2+2\times3+2\times3^2+2\times3^3+2\times3^4+2\times3^5+2\times3^6+2\times3^7+2\times3^8 = 3^9-1$$

and a corresponding symbolism in the other bases. An attempt to generalize symbolically using 'code letters' failed absolutely at this point; the children clearly had not had sufficient experience from which to generalize. Likewise interpreting the experience as representing powers of a fraction was not acceptable.

Then it was suggested that only one piece could be taken at a time in base 3. A tower was built right next to the one with two each, but the children did not realize that it contained only half as much wood as the more substantial tower. The towers looked too different from each other for the relationship between them to be apparent. The two series were then laid out in two horizontal rows, with the corresponding terms paired, and they realized that the row of blocks consisting of one piece a term was half the amount of the row consisting of two pieces a term. This was soon symbolized into

$$1+3+3^2+3^3+3^4+3^5+3^6+3^7+3^8=\tfrac{1}{2}(3^9-1)$$

The same was tried with the other bases, the result coming almost immediately. The generalization that a division by 1 less than the base was necessary was also not long in making its appearance. It was later quite quickly accepted that it was possible to use X instead of a particular base number, and p for the highest power reached in the row; the division now had to be carried out by $X-1$, and the power which was 'almost' constructed when $X-1$ pieces were used was the $(p+1)$th power. Very little guidance was needed to have the discoveries made explicit in the formula

$$1+X+X^2+X^3+\ldots\ldots+X^p=\frac{1}{X-1}(X^{p+1}-1)$$

Then the following were tried: 'An army fights a battle, but they do rather badly and half the soldiers get killed. But the remainder are very brave and fight again, but again half of them get killed. They still will not give up and fight again, but the enemy are too strong and still half the fighters get killed. They fight altogether 8 battles, and each time half the fighters get killed. At the end of all the battles only one soldier is left. How many soldiers were in the army before the first battle?' This and other problems were satisfactorily done making effective use of the formula, and identifying the armies with the blocks on the table, and the last soldier with one unit cube. In other words, the isomorphism was established not between the problem and the formal structure represented by the formula, but between the problem and the properties of the blocks.

At a later session the children were given some triangles and trapezia and asked to make a series 3. As soon as they realized that three of the trapezia could be used to make an equilateral triangle, they constructed the series as far as the largest possible power that could be reached with the material available. They were then told to make something putting two pieces together each time. Tom said 'I know there is going to be a bite left out of it,' and great shouts of 'I knew it! I knew it!' greeted the confirmation of this prediction. Clearly they had not only generalized to an algebraical formula; the formula already represented something abstract, as they were able to recognize the structure both in another embodiment and in practical examples. Transfer had been established.

There was very little difficulty in Group E about forming isomorphisms. Most of the children took it for granted that it was just the rule-structure that made the game what it was, and that two games with different embodiments but with the same rules were 'really the same game'. For example in the transformation game, which succeeded the group-theoretical games on which this structural insight had been obtained, a number of the children would use squares and triangles as vectors and coloured markers as 'counters', or reverse the roles of the markers and the squares and triangles. There was some confusion at times, however, between the component pieces of the vectors and the counters used to 'measure' them. It may be that the self-generated noise of purposely confusing the roles of the counters and the vector components might have helped these children to separate out the cycles

which differed (in one of them the actual vectors were the subjects of the operations, and in the other the numbers of each kind of component). This is not to say that self-generated noise always has beneficial effects; it probably mostly has a detrimental effect on the abstraction process. In an ideal situation the teacher should control the amount of irrelevancy each child has to cut through, so that the process is neither too easy and therefore boring, nor too hard and so apparently hopeless.

4.4. *Abstraction and syntax*

Just as there was a marked difference between Groups D and E in the children's ability to see common structures in the games, so there was a corresponding difference in their readiness to use the matrix of the games as a way of answering questions about them. The children in Group D would try out the moves of an actual embodiment of a game to answer questions such as whether a game was commutative or not, whereas the children in Group E would automatically look at the matrix and read off the reply. The readiness to symbolize and to interpret may indeed go hand in hand with the ability to abstract. It is difficult to talk about a mathematical structure if there is no language in which to talk about it, and it is hard to see how a new cycle of abstraction can be begun without at least the rudiments of a symbolism in which to recapitulate the findings of the last cycle. But we shall come back to the function of symbolism in mathematical thinking later on.

An unexpected difficulty came up when learning the distributive law of multiplication. One subject, having successfully generalized for two variables from the concrete experiences which had been planned for her, said: '3 times F and 3 times S is 3 times F and S. That seems so obvious!' Another subject, having generalized to three variables, and having written down

$$N \times F + N \times S = N \times (F + S)$$

appeared bored and declared: 'This card doesn't teach me anything!' Another subject commented acidly: 'Why did you distribute the distributive law?'

It seems that the distributive law states something whose mathematical content is somewhat trivial, and children find it difficult

to see what there is to learn about it since they appear to know it already. It seems to them to say that the parts put together make a whole, which they no doubt regard as obvious. The importance of the distributive law is perhaps more syntactical than mathematical; it prescribes a correct way of passing on some mathematical information. Parentheses are necessary to avoid misinterpretation of the mathematical communication; they form an essential part of the mathematical language rather than of what the mathematical language speaks about, which is mathematical structure. It is interesting to note that there was no such feeling of redundancy with the commutative law. This may be because life itself is on the whole non-commutative; it is usually quite important which of two things we do first, for example we cannot eat our dinner before it is prepared. Therefore if multiplication is presented to children in situational form, as it was to the children in Group A, they are likely to look at it as a life situation, and so possibly not expect it to be commutative. There may be another more philosophical reason behind the difference. The commutative law states that in multiplication the operator and the operand can be interchanged. This is really rather surprising; the distributive law keeps the operator and operand distinct. In one form when we say

$$\text{N times (A+B)} = \text{N times A} + \text{N times B}$$

we are splitting the operand; in the other form when we say

$$\text{(N+M) times X} = \text{N times X} + \text{M times X}$$

we are splitting the operator, but their respectives roles are not altered from one side of the identity to the other. In this sense the distributive law is logically simpler than the commutative law, and this may be the reason why it is regarded as so obvious.

If there is a difference between learning the distributive law and learning the commutative law from the psychological point of view, it is unlikely that this is an isolated example; it is more likely to be a part of a more general pattern. It could be that different parts of mathematics are inherently more syntactically or more structurally loaded. To a grammarian the distinction might sound absurd; he needs to remember that structure *is* the content of mathematics, and the syntax or even mathematical 'spelling' are necessary in order to communicate parts of the structure from one person to another—this is,

after all, if not the origin, certainly one of the principal functions of any language. There may of course be other differences in the body of mathematics which induce highly differential reactions. I made an attempt once to sort out mathematical theorems according to the degree of rigour employed in their proof.[1] The degree of rigour depended on the numerousness of the universe of discourse to which the existential operators 'there exists x such that' and 'for all x such that' were permitted to apply. The wider the universe, the less rigorous the argument. Some cataloguing of mathematics from the logical, or, better still, from the psychological point of view, is long overdue. Without it, statements such as that one theorem is harder than another one have no ascertainable basis of fact. This is why dissecting the relationship between mathematical structure and the mental processes of acquiring these mathematical structures is needed urgently before substantial advances can be made in the field of mathematics-learning.

4.5. *Construction in mathematical thinking*

Before we pass on to the more logical aspects of mathematical thinking we must say a few words about the constructive aspect of such thinking. Constructive thinking takes place when one aims at a set of requirements and attempts to build a structure which will meet them. We know the kind of structure we are aiming at and somehow, without quite being aware of all the internal intricacies we are constructing, we manage to build a structure with the stated requirements. An artist does essentially this kind of thinking; he does not know exactly what picture he will have painted before he has painted it but he has a definite idea in mind of the kind of picture he wants it to be.[2] He appears equally certain when he does not succeed in his attempt to measure up to his own standard, although he would probably find it difficult to put his reasons into words. Mathematical constructions eventually arise out of mathematical manipulative play; what we call one construction is largely arbitrary, as practically every construction can be built on to as well as taken down from, and it may be difficult to say where one construction ends and another begins. Abstraction is essentially constructive in character. We start with elements and eventually build them up into a class by becoming aware of the defining

[1] 'Sui gradi di rigore', *Rendiconti Seminario Matematico* (Torino, Vol. II, 1951).
[2] See Sir F. Bartlett. *Thinking*, Allen & Unwin (London, 1958).

attribute that must, perhaps not very consciously, have induced us to class the elements together in the first place. Forming a class is only one way of constructing mathematical structures. Constructing any mathematical entity which has certain required properties is constructive. For example, we might be working in the theory of functions of one real variable; if we take an infinite sequence of continuous functions, the limiting function, when it exists, may or may not be continuous. If it is not continuous, we can call it a discontinuous function of class 1, the continuous functions belonging to class zero. One such function would be the limit of the functions x^n taken between 0 and 1 inclusive, whose limit as $n \to \infty$ is the function which is zero between $x=0$ and $x=1$, but is equal to 1 for $x=1$. We then wonder if there are discontinuous functions that are not limits of continuous functions; and if there are, if they are all limits of functions of class 1. The latter could be given the class 2. We might for example ask whether there are any functions of class 3? We could answer showing that it would be impossible for such functions not to exist, by examining the analytical aspect of the idea of function in general. This would be an analytical solution and not a constructive one. A constructive solution would aim at actually constructing a function which would be a limit of functions of class 2, not itself of class 2 or less. A function of class 2 is, for example, one which assumes the value 1 for all irrational values and the value zero for all rational values. The reader might find it amusing to try to prove this. The proof would of course be the analytical part of the construction which had to take place before finding that this particular function belonged to class 2. Another example of construction is the Peano curve which is a continuous function passing through all the points of a square. The requirements of continuity *and* of passing through all points of a square are met by constructing such a function.

Complex 'working to specifications' are of course not to be found in our experimental groups' protocols; the constructions found there are mostly abstractions. There were the makings of the inequality concept in the Group B protocols quoted earlier, which is quite a complex structure, but it is doubtful whether during the period observed any of the children developed such mastery over this structure that we could say they had built a mathematical construction. This might have been because the children were being forced into too wide a front of thinking and were unable to cope with such a width of abstraction.

The term concretization was used earlier as the reverse of abstraction, although I pointed out that it was impossible to un-abstract something. The process can only mean a kind of checking whereby one can ascertain whether a subject has developed a certain breadth of abstraction. The transfer tests were based on this process and we assumed that if a child could concretize, he must have abstracted first. But the whole question is so intimately connected with symbolization and language that a full discussion must await the unravelling of these processes.

3

Generalization and analysis

5.1. *Generalization*

Generalization, as we have said before, is an operation between classes, namely the extension of a class from a primitive to an extended class. This is what I have called elsewhere[1] primitive generalization, in which a certain class A is extended to a class B which includes class A. The more sophisticated kind of generalization, that I call mathematical generalization, is establishing an isomorphism between one class and a sub-class of another class. A primitive generalization is, for example, to pass from a cups-boxes space to a cups-boxes-gloves space. A mathematical generalization is to pass from a cups-boxes space to a North South–EastWest–UpDown space, because the cups-boxes space is isomorphic to the NorthSouth–EastWest space, this being a sub-space of the NS–EW–UD space.

The most frequent form of generalization is extending the class of cases where a certain thing can be done from a finite to an infinite class. When children learn that they can do a certain operation for particular values of the variable, there comes a moment when they suddenly realize that the value of the variable is irrelevant and the operation may be carried out for any value of the variable. The finite set of values of the child's own experience of the variable suddenly extends beyond even his potential experience when he finds that the operation applies whatever value of the variable is chosen. The 'chosen number', the variable, acquires an existence of its own, and the rule governing the operation is now more general than before.

Another type of generalization is from an infinite class to another infinite class, still called primitive generalization if the wider class actually includes the primitive class or mathematical generalization if

[1] Dienes, Z. P. 'On abstraction and generalization', *Harvard Educational Review* (Summer, 1961).

it has a sub-class isomorphic to, but not identical with, the primitive class. This can take place either formally through the symbol system, or in the field of actual experience. In the case of formal generalization it is not self-evident that the generalized concept can be interpreted, as the generalization may only be a formal extension of the symbolism—without any regard to what such extension might mean in relation to actual situations. Examples of all these kinds of generalization will be given from the protocols. It seems that generalizations on a narrow front, i.e. within the framework of the activity in hand, are not found difficult by children, although they find it difficult to accept symbolism if the front is too narrow. An instance of this is Example 15, where the subject resists the $L \times L$ symbolism, whereas she does not resist the $4 \times L$; the square concept had not been broadly formed, but presumably the concept of multiplying by 4 had an adequate breadth for it to be symbolizable.

An example of a generalization on a narrow front, just using the balance, is given in Example 16. The subject wrote down the generalized rule in a particular form as well as stating at the end of the session that 'you go up one' (but only after some errors such as 'you go up two each time' earlier in the session). Progress on such a narrow front is somewhat tenuous, and it is very easy to lose the thread. Probably the difficulty this subject had in accepting the $L \times L$ symbolism was lack of *abstraction*, and not her inability to generalize. The abstract idea of 'as many groups as there are things in a group' had not been formed, and so its concretized form of 'as many rows as there are unit squares in a row' was not available for verification, and its symbolization into $L \times L$ not acceptable.

Generalization as a simple inclusion relationship appears to be difficult at the nine- and ten-year-old level. There were only isolated cases of insight in Group D of the inclusion of one game in another. One subject picked out the two 4-move games from one of the 8-move games. He identified the first 4-move game without help, as well as a 2-move game of 'half-spin and still', but needed some hints before he identified the second 4-move game. Although abstraction was also found difficult by these children (i.e. they found it hard to identify games by rule-structure), such insights were more frequent than logical insights like inclusion.

A generalization from one infinite class to another infinite class follows in Example 19 from the Group C protocols:

Example 19 Group C Fractional logarithms

At one session the children came in to find the series to bases 2, 3 and 5 laid out in order, with the 'place numbers' chalked by them. The series to base 3 was laid out near the blackboard, increasing from left to right, and with the following matrix with the spaces empty for the children to fill in:

Base 27	0	$\frac{1}{3}$	$\frac{2}{3}$	1	$1\frac{1}{3}$	$1\frac{2}{3}$	2	$2\frac{1}{3}$	$2\frac{2}{3}$	3
Base 9	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$
Base 3	0	1	2	3	4	5	6	7	8	9
Piece	1	3	9	27	81	243	729	2187	6561	19683

The only spaces that were not empty were the ones in the first column where the base numbers 27, 9 and 3 were written in. The numbers in the bottom row were represented by the blocks corresponding to each number. The actual numbers were not written in.

We explained that the word logarithm, log for short, was used for the power to which you had to raise a certain base to get a given number. We gave some practice in using the new terminology, calculating logs of the same number to different bases (for example 64), and the children soon associated the position numbers of the blocks representing the numbers with the logs of the numbers. They also quickly realized that there was no sense in talking about the log of a number without specifying to which base. The children were then asked to fill in the table on the blackboard, putting in all the logs they could. They quickly filled in all the whole numbers that are in the above table. Don noticed that for the pieces with 729 units, $2 \times 3 = 6$, but unfortunately there was not sufficient evidence at this stage to exploit this insight. Then Carol said that the empty spaces for the logs of 9 ought to be filled with halves, and proceeded to fill in the table. The thirds and two-thirds on the base 27 were not long to follow, and remarks like the log of 1 is zero in all bases were treated by the rest of the children as obvious tautologies.

A similar table was filled in by Don with the series to the base 2,

where bases 4, 8 and 16 acquired their integral as well as fractional logs. We then felt that it was time for feed-back, as the fractional logs had been discovered as a mathematical extension suggested by the symbolism, and it was time to interpret this discovery in terms of the concrete reality present in the block material. The children's attention was directed to the row of cubes built up on another table. Logs to the base 8 were written on the table by the corresponding structures, and then one of the children enquired if there could be some fractional logs there. First they suggested a square shape that went between the two successive cubes, but soon discovered that in that case the square shape would be 4 times as big as the smaller cube, and the larger cube would only be 2 times as big as the square shape. The introduction of the square shape spoilt the series. It was not long before they introduced the other shapes, and put fractions with denominators 3 as the logs of the new shapes to the base 8. Then we focussed attention on the shape which had $\frac{2}{3}$ for its log; it consisted of 4 unit cubes in the shape of a square. How was it possible to get to this shape from the base, which was 8? Don said you halved 8, then you got 4, but he was told to look at the blackboard, where 9 had a log of $\frac{2}{3}$ to the base 27, and 9 was certainly not half of 27. He did not, however, realize that 9 was one-third of 27, which was really fortunate, because this insight would have led most naturally to division by the cube root of the base, which in turn would have eventually led to $\frac{2}{3}$ via $1-(\frac{1}{3})$, and this procedure would no doubt have been harder than the one suggested by Ruth. This was that you could take the cube root and then square. This would work in the base 27 as well as in the base 8. When asked how what she had said was to do with the $\frac{2}{3}$ which they had written down as a log of 4, a beatific smile of recognition came over Ruth, and she said excitedly that of course the 3 in the $\frac{2}{3}$ was the cube rooting, and the 2 was the squaring. There was unfortunately not enough time left to exploit this insight, although one or two checks were made to see if this newly found rule worked for other fractional logs. This is the kind of insight which is of extreme importance in learning mathematics—the real dovetailing of a symbolic structure with a structured experience. In this case the formal generalization (from integral to fractional logs) had taken place first, and it was the possibility of interpreting this generalization in a real situation that closed the cycle. This was the purpose of the feed-back to reality, and this purpose was clearly achieved.

There are some cases of extremely rapid generalizations in the Group A protocols; some appear to be really rather hasty ones from the mathematical point of view. Take the following example; one wonders if the extremely rapid formulation of the result as

$$(H+4)\times H+(3\times 1)=(H+3)\times(H+1)$$

was the result of a generalization or simply the result of complying with a request to write down the numbers in terms of the hook numbers that were chosen on the balance. When asked whether it would work for another value of H, namely $H=3$, however, the answer was quite definitely in the affirmative, and the 'you need one more' in the response referred to the fact that one more washer was needed to balance the other side. This is mathematically quite correct *and* general.

In his solution of Card C the subject appears to argue that if something works for even numbers and for odd numbers, it works for all numbers. This is a neat example of partially extending a particular result into a general result before extending it wholly. Of course the way the argument is used is incorrect, since only one even number and only one odd number were tried. This readiness to generalize hastily may have been due to the training these children received during the sessions, where practically all the hypotheses they made were eventually confirmed either by further similar experiences or by appealing to the authority of the experimenters. There were not really any contrast cases which would have put the children on their guard against hasty, even if correct, generalizations.

Contrast is a very powerful stimulant to thinking. The proverbial fish that will never discover water is present in all thought processes where contrasts are not provided. Black stands out against white; black against black is entirely shapeless. It is well to remember that this applies even more forcefully, if possible, in the field of cognition. We shall return to the importance of contrast in the section on educational implications.

Example 20 Group A Quadratic factors on balance

Special Card

Put out $(P+4)$ cups of pegs each. Then put out another 4 pegs. Can

you rearrange them into as many cups as there will be pegs in each cup? How many cups will you need?

Write down in your book

$$(P+4)\times P+4=(\quad)\times(\quad)$$

filling in what should go in the brackets.

Now try (P+6) cups of P pegs and 9 extra pegs.

Special Card

On the left put 4 more rings on a hook than the number of the hook. Then put 3 rings on hook 1, also on the left.

Balance all this on the right, using only one hook.

How many ways are there of doing it?

Let us use a code letter for the number of the hook. It could be H. Then you have put (H+4) rings on hook H, and 3 rings on hook 1.

Write down in the language of algebra how this is balanced on the other side.

How is what you have done on the balance to do with what you have done with the cups and pegs?

Try now to make up some of your own that will always work, whichever hook you begin at.

S. '4 pegs in a cup, 8 cups.' (He took 3 extra pegs according to instructions. D explained to him how to do it) 'Like the schools in Patagonia.'

D. 'Somewhat.'

S. (To O) 'Don't write it down, what I'm thinking. I know exactly, take 1 away.'

D. 'Now do it with 10 cups, how many pegs in a cup?'

S. 'You can choose your number.'

D. 'No.' (He explained why not)

S. '7 cups, 5 pegs in a cup.'

D. 'Have you decided how many pegs per cup?'

S. '5.'

D. 'Then how many cups?'

S. '9.' (He put 5 pegs in each cup, and took 3 extra pegs)

D. 'See if you can solve the problem by emptying 1 cup only.'

S. '10 cups, 6 pegs in a cup, and 3 extra pegs.'
D. 'Can you empty one cup?' (S leaned back and thought)
S. 'Yes.'
D. 'Can you always solve this problem by emptying 1 cup?'
S. 'Yep.'
D. 'What's P?'
S. '6.'
D. 'P is the number of pegs in one cup, so there are P+4 cups. How many are there when you empty one?'
S. 'P+3.'
D. 'Will it always work?'
S. 'Yes.' (We gave him another card. This is also a special problem for this subject)
D. 'This one is on the balance.'
S. 'Good.' (S made another rattlesnake and had it 'bite' D. He looked at the card) 'It's the same thing. It'll work for any number.'
O. 'How is it the same?'
S. 'Never mind, I want to do the problem first.' (S put 9 rings on hook 5; the elastic broke)
S. 'I'll make it 2, 6 on 2. It'd better hold or I'll quit.' (He grinned) 'We want 15, on the right.' (He put 3 rings on hook 1 and 2 rings on hook 6)
O. 'That's not what the card says. You can only use 1 hook.'
S. '15, 5, put 3 on 5. It balances.'
Sam (To S) 'How did you figure it out? Suppose I'm very stupid, explain it to me.'
S. '2, 4, 6, put 6 on 2, count to 12 by 2's. 2, 4, 6, 8, 10, 12, put 3 on 1, that's 3, see, stupid?' (On his desk S wrote):

$$(H+4)\times H+(3\times 1)=(H+3)\times(H+1)$$

D. 'Would it work for any H? If H is 3, for example?'
S. 'Yes, but I don't want to do it. $7\times 3+1\times 3$, that's 24, you need one more.'
D. 'How are these two problems alike?'
S. (jokingly) 'They're both written down on cards.' (It was time to go, so S got up. When he got to the doorway he turned around and said, 'And they're both typed, and I can think of one way they're

different. One of the cards has another problem on the back, and one doesn't.' Then he left)

Special Card

Try to find out what goes in the missing places:

$$(A+5)\times A+6=(A+\quad)\times(A+\quad)$$

If you put 3 less rings on a hook than the number of the hook, as well as 2 rings on hook 1, all on the same side, how could you balance all this on the other side, using only one hook?

Can you write down this problem in the language of algebra? Try. When you have done this problem, write down in algebra how the one side is equal to the other side.

DO EACH PROBLEM AT LEAST TWICE, THAT IS, STARTING AT LEAST IN TWO DIFFERENT PLACES ON THE BALANCE.

S. 'Do you still want me to tell you how they're alike?' (The two problems he's been working on this week. S looks at the two cards)

S. 'They're the same problem only you use different things.' (D gave him the next problem card)

S. 'Is A the code number? I know what it stands for, it stands for any number. I'll choose an A and then solve it. My A is 3. $3+5$ is 8, $8+3$ is 11, $+6$ is 17. No, 8×3, not $8+3$. 8×3 is 24 and 6 is 30. 30 is A times something? Now to think of that darn something. Do the somethings have to be the same?'

O. 'No.'

S. 'Can they be the same?'

O. 'If it balances.' (S leaned 'way back in his chair so that he was nearly on the floor. He closed his eyes) 'This is boring. 3×10.'

O. 'How else?'

S. '2×15. $A\times10$.'

O. 'Can you do it so both of your numbers are less than ten?'

S. '6×5.'

O. 'How can you write that, using A?'

S. 'Well, 5 is $A+2$, $A+3$ is 6, but it won't work for A is 4 or 5 or 6.'

O. 'Try it for 4.'

S. '$4+5$ is 9. 9×4 is $36+6$ is 42. $4+2$ is $6\times(4+3)$ 6×7 is 42. Then it works for all numbers.'

O. 'Why?'
S. 'Because it works for an even and for an odd number.' (D began the next part of the card)
D. 'Put 3 less rings on a hook.'
S. 'Use hook 3, don't put any rings on it.' (S went on) 'Put 2 on 1 and 1 on 2. It balances.'
D. 'What about using hook 4?' (S put 1 ring on 4 and 2 rings on 1 on the left, on the right he put 1 ring on hook 6)
D. 'How much more is that than your number?' (No response from S)
D. 'How much less is the number of rings used?'
S. '3.'
O. 'Do you think it will work for 5?' (S was playing with my pen. He twirled rings on it)
S. 'I don't know, I give up. Want me to shoot a peg without touching it?' (He used my ball-point pen like a billiard cue)
O. 'How would you balance this if your number was 9?'
S. 'Put 6 on 9.' (At this point C asked, 'Where's my ruby-headed eraser?' There was chaos for about 5 minutes, a great free-for-all in which everybody joined. When S finally got settled down again, he said) '6×9 is $54+2$ is 56. I'll have to burglarize some more rings.' (S put 8 rings on hook 8) 'That's 56, no, it's 64, put 8 on 7. This is boring, I think this is dull.'

End of session on the following day went as follows:

D. 'Call H the number of the hook, or call it N; N is 9 here.'
S. 'Call it oogly-hoogly. So that's oogly-hoogly minus 2. Then the number of pegs is oogly-hoogly minus 1 so oogly-hoogly times oogly-hoogly minus 3 plus 2×1 is equal to oogly-hoogly minus two times oogly-hoogly minus 1.'
D. 'Let oogly-hoogly be 8.'
S. '42, it works.'
D. 'Let's write it down; let oogly-hoogly be U.'
S. 'No, UH.' (D wrote (UH minus 3) $\times$ UH $+2$),
S. 'That's (UH minus 2) $\times$ (UH minus 1).'

(S wrote this down. He took the eraser, fell on the floor and began a mock fight with R. It was time to go)

The end of the report is an example of a higher-order type of play than the instances in the first chapter. It most likely resulted from the mastery of a generalization in the abstract. This subject no longer 'plays' with the material, but 'plays' with an idea he has extracted from the material. This idea is that any kind of symbolism will do to describe the mathematical identity discovered. The subject is playfully glorying in this freedom, the humorous name 'oogly-hoogly' giving expression to it. It is quite possible that this complete mastery and ease of manipulation of the idea had not been achieved in the previous session, at the end of which he says, 'I don't know, I give up,' and 'This is boring, I think this is dull.' The joy of really closing the learning cycle showed itself in humorous manipulation of its component parts. This example suggests a line of enquiry on the connexion between humour and learning which may pay good dividends in educational theory and practice.

5.2. *Restriction*

There are only a few examples of the reverse process of generalization, that of restriction or particularization, except in formal situations where particular values are given to variables that had been generalized. One whole session in Group D was devoted to restrictions in the vector-space game. At first a restriction was placed on the kind of vector it was possible to choose; then this restriction was lifted and another one was imposed. Finally both restrictions were imposed simultaneously, leading to two simultaneous equations in a two-dimensional vector space that yielded just one solution.

Such an exercise requires at least an implicit appreciation of the logical inclusion relationship. The kind of cups and boxes you are allowed when the rule has to be obeyed is a class included in the class of all possible arrangements of cups and boxes. The difference between this and generalization is that you pass from the more extensive class to the restricted class, whereas in generalization the passage is in the opposite direction. It comes out quite clearly from the protocol that a restriction involving adding only was less difficult than a restriction involving multiplication. It is very likely that a restriction involving the multiplication of variables would have been even more difficult; for example, a harder restriction of this kind would be that the number of cups multiplied by the number of boxes had to yield 36 dead matches, with the following 'match' multiplication rules:

live × live = live, live × dead = dead,
dead × live = dead, dead × dead = live,

or of course any other arbitrary multiplication rules would serve. These rules are isomorphic to multiplications of positive and negative numbers, where positive numbers are live and negative ones are dead. For example 4 closed boxes and 9 upside-down cups would have complied with the above restriction, since 4 live × 9 dead = 36 dead.

This restriction exercise is clearly harder than, say, the introduction of an extra dimension, which is a generalization. A remark by one subject, 'This is tiring', shows that the children were being considerably stretched in these exercises.

Example 21 Group D Sub-spaces

D. 'You've worked on some problems, now we're going to work on some related problems. Some are pretty tough. We're going to need a language.' (Then D wrote on the board)

right-side-up cup —live matches

upside-down cup —dead matches

closed box —live matches

open box —dead matches

B. 'Make yourself some open boxes. Imagine that there are different kinds of matches. Can you show me how to lay out some of these? Sandy, tell me a rule.'
S. 'An upside-down cup and a right-side-up cup.'
B. 'That would mean a live match and a dead match.'
S. 'They don't count.' (Collette looks puzzled)
B. 'They will cancel each other out.'
C. 'Like opposites.'
B. (To L) 'Try to lay out things so that you would have 9 of the same kind of matches.'

Larry, Sandy, Jon and Collette started to work. Paula did not begin to participate. She did not seem to understand. D showed her what it was all about.

J. 'That's a cinch. 9 dead matches.'

B began to record the rules on the board; he wrote out Larry's rule.

Larry's 6 +6 +9 =9 dead matches

Then Jon's and Collette's 10 +1 =9 dead matches

B. (to L) 'Use all the materials, all the different kinds of things.'

Paula in the meantime had worked out a rule as follows:

5 +5 +9 (mixed solution)

J. 'This is tiring.'

B. 'Here's a real tough one for L. Fix it so that you have twice as many matches in the cups pile as in the box pile. Make them the same kind of matches.' (He gave the same problem to J and S, after checking the earlier solution to the cup and box cancellation problems. C and P were given the same problem)

LARRY

At the beginning L was quite confused by the problems. He thought that he had to have 18 in one pile, working with the 9 that he had, 9×2 equalling 18. He attempted to get 18 dead matches in one pile using boxes, and 9 in the other, using the cups. B explained to him again what the problem entailed.

L. 'That's easy. If I had 0 in 1 pile, then twice 0 is 0.' (Then he set up 2 open boxes and 2 closed boxes on one side, and 2 upside-down cups on the other. B picked him up on this and said that he was right, that 2×0 was 0 but also 16×0 was 0 and 18×0 was 0, so the solution wouldn't work. L then set up 15 open boxes on one side and 6 closed boxes to make 9 dead matches, and, on the other, 20 upside-down cups and 2 right-side-up cups, again for 18 dead matches)

B. 'You have to end up with 9 dead matches with twice as many in one heap as the other. $9+18$ does not equal 9.' (Then B put this frame on the board)

$$\text{——} + \text{——} = 9$$

B. 'Now, what numbers go here, so that one would be twice the other, and still equal 9?'

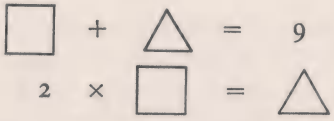
L. '8+7 then 5+4.' (Finally he got 6+3. With this he easily worked out the solution to the problem. It was here that L showed that he couldn't handle two variables at one time)

JON AND SANDY

These children had much the same trouble with the problem involving twice as many matches in the cups piles as the box piles. They approached it much the same way as L. First they had a number of boxes cancelling out, plus 9 closed boxes on one side and 18 cups on the other. L showed them his solution which involved 6 upside-down cups and 3 open boxes. They seemed to understand this.

PAULA

P approached the problem setting up, according to Bill's notes, 4+6= 10, saying, 'One has to be 2 more than the other, and these add to 10.' She too had trouble with the 3+6 equalling 9. Bill notes that perhaps telling them, or showing them



would have led them on faster to a solution. B gave Paula and Collette a problem of arriving at 10 live matches, using cups, small boxes and large boxes. They worked through this problem, ending up with 8 closed big boxes, 4 closed small boxes, and 2 upside-down cups for a total of 10 live matches.

Special note: Ross was absent today. D feels that he perhaps might have been the real disrupting element in the group, since today the children worked together quite effectively. Even L settled down and worked fairly consistently throughout the period.

6. *Other forms of analytical thinking*

It should not be forgotten that generalization and restriction are still rather special kinds of logical processes. There are other relation-

ships that can exist between classes besides inclusion. One is identity, corresponding to the mental process of forming an isomorphism between the elements of two classes. In this case we can say that, at any rate as far as the operations considered relevant in the isomorphism are concerned, the two classes are for our purposes identical. Classes deemed identical in relation to one kind of operation may not of course be identical in relation to other operations. It is important here to distinguish forming isomorphisms between structures just in the process of being developed from isomorphisms between structures already formed. The first is a constructive process, and has been termed abstraction in these pages; the second is not constructive because the structures are already constructed. Establishing the isomorphism here leads to the realization of the logical relationship of identity between the two classes. It is in this sense that forming an isomorphism is logical or analytical.

The relationships of overlapping and distinctness can also exist between classes. We made no attempts with any of the experimental groups to induce the mental processes that would result in the realization of these two kinds of relationships. Logical relationships appear to be difficult for young children to grapple with. According to Piaget one should not expect logical thinking at all during what he calls the concrete operational stage of development. This stage passes over into the more adult, logical stage round about the age of twelve, and it is true that after this age the logical-analytical aspects of structures (their internal relationships and their relationships to one another) are more easily appreciated. There is, however, no reason to believe that younger children could not be trained to think in terms of logical categories rather than mainly constructively. It would seem *a priori* probable, prevalent theories notwithstanding, that a series of practical experiences where the logical structures are easily discernible and where it is necessary for children to realize them to solve given problems would accelerate the growth of children's logical abilities.[1] There are certainly signs that if children's experience is saturated with mathematical structure they will ask questions of a logical character after a certain time. This is presumably an attempt to sort out the

[1] Inhelder has been working at Cognitive Center, Harvard, on this and related problems (1961–2). See also William Hull's 'Concept work with young children' (*Bulletin of the International Study Group for Mathematics Learning*, April, 1963).

mathematical structures that are piling up in their experience, and the efficient processing of all this surfeit of information eventually compels children to look around for ways of ordering all the information.

During a fourth-grade algebra lesson a number of children were having concrete experiences equivalent to the mathematical identities

$$x^2+4x+4=(x+2)^2$$

and

$$x^2+6x+9=(x+3)^2$$

There was a group of boys and a group of girls working on these exercises. Both groups wanted to generalize the experience to $x^2+8x+?=(\quad)^2$, $x^2+10x+?=(\quad)^2$, and so on. With a little perseverance, both groups eventually found what they called 'the pattern'; namely that half the number of x's had to be squared, and then you would get another square. The difference between the girls and the boys was striking. The girls went up to the boys very excitedly and told them that they had discovered the pattern and could now do them for as large numbers as they chose. One of the boys appeared very superior about this and declared to the girls:

'We have found the pattern, of course, but what we are trying to decide is why it works!'

The boys were immediately opening the closure of 'the pattern', whereas the girls were quite content to stay within their new-found rules and play according to them, never once wondering what made them just those rules and not some others. A short exercise with the cups and beans convinced the boys that at any rate x^2+4x+4 always had to be the same as $(x+2)^2$, and they were very excited to have proved their first mathematical theorem. It is probable that they would not have reacted like this had they not fully understood the mathematical structure. Here, as once before, children started on a second cycle within one lesson.

With an act of generalization (as with an act of abstraction) comes a sense of power, and with this a sense of exhilaration. This is where the energy supply is tapped for further and more difficult enquiries; success is very important, and success breeds success, much as failure breeds failure.

I can recall another fourth-grade class in which many of the children were working with the MAB. The task was to construct a cube with one $N\times N\times N$, three pieces $N\times N\times 1$, three pieces $N\times 1\times 1$, and one

unit cube $1 \times 1 \times 1$. The discovery that this could be done independently of N (i.e. whichever box (base) was chosen), *and* that the pattern worked out in exactly the same way, was greeted with enormous excitement; it appeared hardly credible that what seemed such a complicated structure could apply to all possible sizes. It is a neat example of a very narrow, though entirely successful, generalization. It would be wrong to conclude that these children had realized the binomial relationship

$$x^3+3x^2+3x+1=(x+1)^3$$

in the abstract. They had learned a general pattern about how certain particular shapes of wooden blocks could be put together to make a cube, and that success did not depend on the size of the blocks but only on their relative shapes. To realize the abstract relationship they would need to appreciate something like the following:

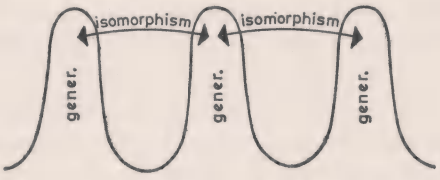
Take a certain size group of any object; take as many of these groups as there are objects in a group; and take as many of these larger groups of groups as there are objects in the first small group. To this add three times as many groups as there are objects in the original group; then to this add three more small groups; then just one more object. This whole collection of objects can be rearranged in a whole group of groups of objects in such a way that each group has one more object in it than the original group. There are as many of these groups as there are objects in each of these groups, and there are also as many groups of groups as there are objects in one of these groups.

It is doubtful if any reader would understand the meaning of this requirement unless he already had an abstract idea of the third power of a number or, if he did not have this knowledge, unless he followed the instructions out to the letter and convinced himself that indeed for any size group this could always be done. The embodiment with which the fourth-grade children were working was a geometrically structured embodiment, where the unit cubes corresponded to a zero dimension, the long pieces to a one-dimensional displacement, the 'flat' pieces to a two-dimensional and the large cube to a three-dimensional displacement. The 'groups of objects' are neatly arrayed along a straight line and can form a 'row'. The 'groups of groups' are the rows neatly placed parallel to one another, forming the shape of a square 'layer'. The 'groups of groups of groups' are likewise neatly arranged in the form of a cube, the 'layers' placed either on top of one another or next to each

other. The eight pieces in the problem can be put together to form another cube, and therefore another group of groups of groups, where the number of unit cubes in a group, the number of groups in a group, and the number of groups of groups in the final group, are all the same; the number is in fact the length of an edge of the final cube measured in our units. This edge is just one unit longer than the edge of the original cube.

A great deal more thought and work will be necessary before this binomial identity becomes abstract in the sense that x can refer to any collection of objects consisting of a certain arbitrary number of objects, called x for convenience. In this case certainly the generalization is easier than the abstraction. The generalization was made possible by the spatial properties of the shapes, not by the fact that the pieces contained units which numbered powers of x.

There is one important problem in connexion with the relationship between abstraction and generalization, and it is important to throw some light on it as soon as possible in the interests of tackling it more effectively in the educational process. It is possible to attack a conceptual structure in a number of ways. One would be to make some narrow and deep inroads into the territory by attempting rather daring generalizations along a number of seemingly disconnected, independent paths. The possibility of establishing connexions between these bridgeheads could then be explored. In other words if we have been fortunate there may be some intimate connexion between our separate generalizations, and perhaps there may be an isomorphism connecting pairs of generalizations, in which case the separate generalizations will turn into one abstract structure. This kind of procedure is represented by the diagram below:



Known structure

Another form of attack, perhaps less daring, would be to proceed rather more slowly towards generalizations, but along a wide front.

One would do this by bearing in mind all the time the many possible ways in which the structure might be applicable, not as a composite (as some of our subjects in Group A had at first conceived) but as an extracted essence. In this case, whatever situation we were using to find the general pattern, we should bear in mind the kind of class of situations our situation is a prototype of. A diagram representing this kind of attack would be a broadly advancing front. Preferring this or the other kind of strategy would be a matter of individual choice or of a stage of development. It is also possible that certain structures are intrinsically more amenable to one kind of treatment than to another.

In an earlier paper[1] I made an attempt at dovetailing mathematical structures in the theory of arithmetic to individuals' attitudes to mathematics on an open-closed dimension. Experimental studies in the validity of such connexions would shed a great deal of light on the problem of strategy. The study of strategies in *A Study of Thinking*[2] attempted to sort out the strategies of different subjects in the same mathematical or rather logical situation. It would be interesting to find out how subjects react differently to a series of mathematical situations graded on some objectively measurable scale, and to gain some idea of the relationships between strategies they select and mathematical structures.

It is worth saying a few words about the position of generalization in the general framework of analytical thinking. Starting from any particular structure, we might distinguish two kinds of analytical thinking: retroactive and progressive. The retroactive consists in analysing the structure that has been made, the progressive in going beyond the structure and attempting to establish relationships of a logical kind between the given structure and other similar ones. Generalization is clearly of the progressive type; it is looking beyond what is known and given—a search for a regularity within the field of operations—or a search for a wider class where the same regularity can be seen to operate. This is what a mathematician does when he wonders whether his result, proved in the case of points on a straight line, can apply to points scattered anywhere in a plane, or in a three-dimensional or n-dimensional space. A retroactive analysis is the examination of a conjecture or hypothesis in a critical way. Experiences in the experimental groups

[1] *La percezione astratta*, Rivista di Filosofia (1956).
[2] Bruner, Goodnow & Austin, Wiley (New York, 1956).

show that an approach to a new problem (with children, at any rate) is always constructive; the analytical aspect of thinking can only come into play when a certain amount of construction has taken place. This is on the face of it painfully obvious, yet there are many adults who approach new situations analytically to start with, before they have had time to size the situation up and see what it is made of. In such cases they really construct the situations by means of preconceived ideas. The analysis will therefore function only in cases which are fairly common and where a standard explanation of the situation is sufficient to describe them. Whenever the preconceived constructions break down the analysis will result in absurd conclusions which are difficult to trace back to the preconceived construction because these constructions are often below the level of awareness. This is why a constructive thinker is usually a more creative and a more original thinker; his thinking pays off in the new and unusual situations, whereas the strategy of the analytical thinker only pays off in the more usual situations for which standard constructions are available.

Some examples follow. The most striking one is from the last of the sessions in Group C when the analysis of the whole powers—roots—logarithm complex was achieved. This is the protocol:

Example 22 Group C Mathematical landscape

In the last session with the children we attempted to establish a view of the whole situation. For the first quarter of an hour exercises were done to familiarize the children with the mathematical meanings of the fractional powers which they had invented by formal generalization. This took some time and effort, but eventually they understood that taking the 6th power of the 12th root was equivalent to taking a square root or to taking the 3rd power of the 6th root. We again used the blackboard, with the pieces standing under the blackboard level with its lower edge. The children were eventually quite able to use the formal matrix of integral and fractional powers to establish equivalences such as the above, and so the formal generalization had been enriched by real mathematical experience.

We then did some inverse exercises and all the children were clear about the inversion of squares and square roots, cubes and cube roots, and so on. Then they were told to write down the power 3^5, and asked

what they had to do to get back to the 3. There was still considerable difficulty over this as the children had their backs to the matrix and apparently the symbolism was not yet meaningful enough for them to turn round and read off the answer. They were, however, able to do it immediately on a series of actual pieces, after which they could also do it formally. Feed-back was still necessary from time to time, as the connexion between the formalism and the experiences from which it had been derived kept breaking. They were asked what they would have to do to get from the 3^5 to the 5. Carol was first to answer that you had to take the log. More exercises of this nature were done, and we established that if you wanted to find the exponent in B^P you took its log to the base B, and if you wanted to find the base you took the 1/pth power (or the p-th root). Before they realized this explicitly in this form, all three types of problem were presented to them in a particular form.

$B^P = R$ was replaced by $3^2 = R$, then by $3^4 = R$ and so on and they had to calculate R in each case. They noticed that this was done by raising to a power. Next, examples of the type

$$B^3 = 125$$

were given. This type caused a little difficulty, and they again had to have recourse to the concrete series before they realized that what was required was the cube root of 125. They made many absurd conjectures, and rejected them as absurd, before this result was achieved. They were given other examples of this type, until they realized that the problem here was solved by taking a root. The children themselves suggested the third type of problem where the exponent was the unknown, and Carol, before the others had time to think about it, said that it was of course taking the log. They did this kind of problem a number of times, and eventually the three operations, corresponding to the three ways of posing the problem, were connected by their true mathematical connexions.

We then digressed into inverses of multiplication and addition, and asked the children why, if you had 3×5 and you wanted to get back to either 3 or 5 you could do so by division, yet if you had 3^5 you had to take a 5th root to get 3, and a log to the base 3 to get 5. We emphasized that the only inverse of addition was subtraction. All the children saw that the situation was somehow radically different but

could not put their finger on the non-commutativity of the power and the base. Then they wrote the equations $2 \times Y = 8$ and $Y \times 2 = 8$ on their desks, and everyone agreed that it was the same equation, in spite of repeated suggestions that it was not. Then Ruth suddenly saw the point that of course

$3 \times 3 \times 3 \times 3 \times 3$ was not the same as $5 \times 5 \times 5$

and that was why you had two ways of doing it in powers and only one with multiplication and addition.

The overall picture (that when you are looking for the power, you are finding a log, when you are looking for the base, you are taking a root, and when you are looking for the result, you are raising to a power) was finally seen by certainly three of the children. Tom was somewhat left out of it. He had missed one of the critical sessions, but apart from that his capacities had possibly been overstretched. All the children exhibited a positive attitude and expressed a desire to go on to parts of mathematics where these newly acquired concepts were used. Don was of the opinion that it was a pity that Ruth and Carol had so often come to conclusions before him as it was much more fun to figure things out by himself and it was a great pity to be told too soon. It was quite clear that all the children had experienced and enjoyed truly mathematical research experiences, and that they were looking to their teachers to lead them to more.

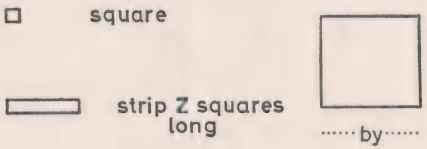
The protocol just quoted is a very neat example of progressive analysis. The abstractions or classifications 'raising to a power', 'extracting a root' and 'taking a logarithm' had been achieved. The object of the last session was to enable the children to see how these classifications related to each other and, as one can see from the protocols, three of the four children achieved this. There was a considerable amount of 'teaching' done in this group; direction of the children's thinking along certain channels was at times quite severe. Whether concrete exercises could have been devised which would have brought the children to the same terminal point with less help even though after a longer period of time remains an open question. It is certainly established that they were able, with help, to reach an understanding of this highly complex structure in a fairly analytical way. The problem remains whether it is worth pushing children along constrained paths or whether to allow

them more freedom to experiment. A great deal of research must be done on the effects of freedom and of constraints of cognitive processes before the position becomes clear.

Here is an example of a subject building up a generalization entirely in one embodiment. The generalization seems firm enough for S to be able to have a measure of analytical insight into the structure. She was able to verbalize the relationship between the number of strip lengths and the number of squares.

Example 23 Group A The square problem

Special Card (1)



Fill in the spaces

Take one large square, 2 strips, and one small square and put them together to form a square. They form a

..... by ..... square.

If the strips were just 8 small squares long, you would have made a

..... by ..... square.

Special Card (2)

Take one large square and a certain number of strips.

Is it possible to make a square with what you have?

Is it possible to make one if you are allowed to have some small squares as well?

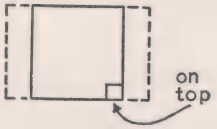
Is this always possible whatever the number of strips was that you took in the beginning?

If it is sometimes possible to make the square and sometimes impossible, can you say what makes it possible or impossible?

Subject looked briefly at Card 1, the new series on squares, which she patently ignored. She started to build 'stairs' with the small squares which had been provided. She did not refer again to Card 1.

S. 'No, because I don't get it.' (Then D began to explain) 'This is going to be hard, such small squares.' (Then she started the following sequence)

1



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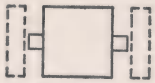

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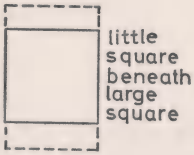
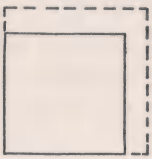


'I don't think I can do it.' (She does not know what to do with the little square)

D. 'What is a square?'

S. 'Same L, length and width, equals an even number of squares.' (Then she laid out)



'I'm not sure if this is a square. No, it's not a square.'

D. 'How long is it?' (S began laying out small squares to find the measure of the large square)

Time: 9.00

S continued to measure the length of the square. When she arrived at the small individual square, she thought it should somehow be divided to get the length of the square.

She apparently thought that the rows and columns should be kept separate, but she did not see that the row-column relationship involved looking first at the rows and then at the columns, and that there were as many units in the rows as in the columns. D then gave her four smaller squares to equal the one small square she had. S wanted to divide the smaller square diagonally to get the length and width of the large square.

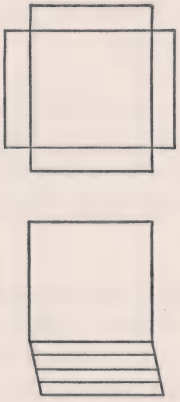
Time: 9.04

S was redirected to Card 1 for the next job which was to symbolize the newly constructed square. S then wrote 'strip length and 1 by strip length and 1'.

D. 'That's a lot to write. Let's use some shorthand.' (S then wrote SP+1 by SP+1 square) 'If the strips were just 8 small squares long, you would have made a . . . ?' (S said immediately 'a 9×9 square.' She then completed the spaces on Card 1)

Time: 9.10

S began work on Card 2. She decided to use 4 strips.



Time: 9.12

S continued constructing. Then she used large squares. (Frustration quite evident at this point)

Time: 9.15

B then suggested that she build some pretty things.

Time: 9.18

D gave her squares and strips. S began to construct.

Time: 9.20

S. 'If you want to get it exactly, it's impossible.'

D. 'How many strips do you want to use?'

S. 'Four.' (D gave her squares along with the strips)

D. 'How long is the square?'

S. 'Strip length plus 2, strip length plus 2.' (D gave her another strip length, 5 now instead of 4)

D. 'Can you make a square?'

S. 'I don't know.' (She tried it using small squares) 'Impossible.'

D. 'Use 6 strips.' (S correctly constructed a square)

(Her construction was excellent. She had the geometry of a square well in mind. D recapped with her)

S. 'You make them with even numbers.'

(Then she started to use 10 strips. She constructed a square and then, when asked for length and width, said immediately, '1 strip length + 5 and 1 strip length + 5')

Then the session ended.

S had the geometry of a square very well in mind. After the initial frustration trying to construct a square with the two strip lengths and the small square, she very quickly achieved insight into the relationship of the number of strip lengths and squares necessary in constructing from the given large square.

To sum up, we can say that narrow-front immediate generalizations are found quite easy by children when this is an extension from a finite class to an infinite class of which the finite class forms a part. The finite class is usually the class of instances in the children's own experience.

There are progressive and retroactive types of analyses, of which generalization and restriction are examples. Generalization is not the only kind of progressive analysis. Gaining insight into the logical relationships between powers, roots and logarithms, and into the concept of the inverse operator, are progressive, and they are not generalizations. Neither is restriction the only type of retroactive analysis. Any kind of internal dissection of a structure already constructed is retroactive analysis. Construction must take place before either of these kinds of analyses can function effectively.

Many problems remain about the relationships between the constructive and analytical functions in general, and between the processes of abstraction and generalization in particular. Research is needed to discover whether, for example, there is a type of person who tends to generalize along narrow fronts and push far ahead and then establish the connexions and so achieve the abstraction in a leap; and another type who keeps the breadth of abstraction fairly wide the whole time. Both strategies may of course occur in the same person, depending on the cognitive task in hand, and much must be learnt about the relationship between structures and optimum strategies.

The relationship between the processes of abstraction and generalization on the one hand, and their symbolization and the interpretation of the symbolizations on the other, will be discussed in the next section.

4

Symbolization and interpretation

7. *The symbolization process*

Man has always used symbols to express himself and to communicate with his fellows. It would be foolish to underestimate the role that the creation of symbols plays in human thinking, although what this role is remains far from clear. A child grows up in a society that abounds in ready-made symbols standing for a number of different objects, classes of objects, situations, acts, relationships, places and so on. In other words he grows up hearing and seeing language. As we go through our experiences, from the earliest ones onwards, they are accompanied by an orchestra of auditory, visual and even tactile symbols (like shaking hands); and one by one we pick out the symbols which appear to indicate some class of events. This is a natural process in the sense that no formal 'teaching' accompanies our learning the mother-tongue, for example. We form our experiences, classify them in order to predict future events, and later use symbols to refer to these classifications. Sometimes we attach symbols to classifications in ways which differ from conventional classifications; we should then say in ordinary language that we have not learned the meaning of a certain word quite accurately. The connexion between the symbol and that which it symbolizes grows progressively stronger and conforms more and more to the conventional classification employed in the culture in which we happen to be growing up.

Let us consider for a moment whether the same situation obtains in learning the language of mathematics. It does not take much thought to realize that there are important differences. One is, surely, that we do not hear around us a constant orchestra of unintelligible mathematical noises; nor do we experience the corresponding mathematical experiences which would enable us to select the right noises to go with certain classes of mathematical experiences. In fact mathematical

experiences, in the sense of experiences whose actual content would be isomorphic to mathematical structures, are almost totally lacking from our lives. This is not quite so in the early stages, when we form the concepts of the separate natural numbers by abstracting from the kinds of objects there are in a group and concentrating on their number. Adding and subtracting are still reasonably common experiences, even division occurs sometimes when we wish to share things out equally. Multiplications, however, leading to higher powers of a base, are practically non-existent in practice, let alone structures like quadratic and higher-order polynomials, complex numbers, definite integrals and so on. And even if they did occur they would not be likely to occur together with the relevant mathematical chatter that would make the situation analogous to learning the mother-tongue.

There is another important difference. In learning our own language there is a fair time-lag between the experiences and their audible symbolization, and an even longer time-lag between audible and visible symbolization. In learning mathematics, it is customary to learn the mathematics, talk about it, and write it, in one and the same lesson. Or if not in the same lesson, the corresponding time-lags are surely much shorter. It is even doubtful whether a distinction is ever made between the mathematics learned and its symbolization. Its symbols are not even symbols, properly speaking, because symbols stand for something which they themselves are not. The content of mathematics is of course abstracted structure, and each structure in some way makes use of some other structure, until we get down to numbers or continuously varying quantities like lengths, which describe collections of objects or lengths of objects respectively. In the long run, mathematics is about the world of things, but it describes some very superficial properties of these things, namely their quantitative aspects. This tends to be forgotten when mathematical thinking is examined; the concrete origin of all mathematical structures should perhaps be part of the mathematics-learning theory which we so sorely need to account for the seemingly very strange activities of mathematicians. In the chapter on educational implications I suggest ways in which such a theory and practice of mathematics-learning may be built up. I shall have to anticipate some of this in this chapter, otherwise I could say very little about the experimental work with the five groups in this study. A great deal of research needs to be done on the process of

transition from the non-symbolic to the symbolic type of thinking. I am only going to make a few tentative suggestions after observing the experimental groups as well as other children who have worked through to symbolic mathematics by means of a series of concrete experiences.

As we have observed in the section on play, mathematical thinking appears to devolve in fairly well-defined cycles. Each new cycle begins by 'playing' with the end-result of the last cycle; the cycle ends when something new (with which it will be possible to play again, and so start a fresh cycle) is fashioned from it. On the whole the ends of these cycles are punctuated by a kind of fixing symbolism which expresses the act of completion in symbolic form. Let us examine the cycles described in Example 1 (page 32) from the point of view of the function of symbolism.

In this example there is really a preliminary cycle before what was termed the first cycle. In this preliminary cycle objects are turned into vectors by delimiting the kinds of objects permitted (choice of the number of dimensions), and by deciding what the opposite positions are going to be, making the convention that opposite objects of the same kind cancel out in pairs. The symbolism here is hardly more than the language used by the subject, perhaps with abbreviations. The letter c might stand for a cup, $\bar{c}$ might stand for an upside-down cup, b for a closed box, $\bar{b}$ for an open box and so on. For example (2c, $3\bar{b}$) would be short for the vector consisting of 2 right-way-up cups and 3 open boxes. The notation $2c+3\bar{b}$ would strictly speaking be premature here, as 'adding' or 'joining' were not really a part of this cycle, and this adding is actually the result of the sum (2c, 0)+(0, 3b). The adding is part of the next cycle and helps to build the vector space.

The first cycle then establishes addition as 'joining' and subtraction as 'removing'. Deciding what you can do with a vector is what turns the vectors into a vector space. Vectors can now be written as (x_1, x_2, x_3, . . .), where x_1 denotes the number of objects present from the first dimension, x_2 denotes the number present from the second dimension and so on. For example in two dimensions addition and subtraction are defined by

$$(x_1, x_2)+(y_1, y_2)=(x_1+y_1, x_2+y_2)$$
$$(x_1, x_2)-(y_1, y_2)=(x_1+\bar{y}_1, x_2+\bar{y}_2)$$

which could be written for short as $X+Y$ and $X-Y$.

The second cycle begins by doing something to the entire vector space, for example transforming it into another vector space. This is done by transformations such as

$$y_1 = a_{11}x_1 + a_{12}x_2$$
$$y_2 = a_{21}x_1 + a_{22}x_2$$

where in the double suffix the first suffix indicates the row and the second suffix the column in which the coefficient is situated in the transformation. After an exhaustive exploration of the properties of these transformations a new 'name' could be given to them, for example we might say that a transformation A is acting on the vector X and the result is the vector Y, and we should merely write

$$AX = Y$$

The third cycle turns the transformation into an inverse transformation. In the notation just introduced, we pass from $AX = Y$ to $X = A^{-1}Y$ where A^{-1} is the inverse of A. It will be clear that if I denotes the identity transformation, i.e. if for all vectors X, $IX = X$, then

$$(A^{-1})^{-1} = I$$

This is already stating a property of the inverse transformation and should really be considered as a fourth cycle which will include identities such as

$$(AB)^{-1} = B^{-1}A^{-1}$$

where A and B are any transformations that have inverses.

The complete symbolism suggested above as cycle-closing symbolism was not introduced fully to the children in Group E, as they did not have enough experience of transformations to be able to close the cycles in the fullest sense. This means that these children went on to further cycles without properly finishing previous cycles. Whether this is good pedagogy or not is an open question; the fact is that a number of children in the group were able to travel fast enough so that at least they had a taste of the later cycles. They appeared to resist the symbolism at times and preferred to solve the problems by carrying out the tasks physically with actual squares, triangles, counters, etc. The carry-over from mathematical experience to the successful wielding of the corresponding symbolism had not been completed, and yet with the

aid of re-enacting the experiences whose later cycles were the generalizations and abstractions, they were able to see ahead to these cycles and even solve problems which would appear very difficult to solve without an effective completion of these cycles. Seeing ahead provided a great deal of motivation for much of the learning that took place in this group. This indicates that the transition from the non-symbolic to the fully symbolized stage may be much longer in the optimum type of mathematics-learning than might be imagined.

The examples that follow are of learning situations which illustrate different stages on the way between the non-symbolic and the fully operational symbolic thinking. How this transition does or should or could take place is a question which will be touched on in the next chapter.

First there are some examples where the cognitive process is still largely at the stage of perception.

Example 24 Group B Cube problem

The following pieces from the MAB set were given to the children: one $N \times N \times N$, three $N \times N \times 1$, three $N \times 1 \times 1$ and one $1 \times 1 \times 1$. Some children were asked to make a cube and some given an $(N+1) \times (N+1) \times (N+1)$ cube as a model to imitate. The children became very involved in this. Many considered the problem impossible at first but as soon as the news spread that somebody had succeeded in making the cube, all the children made great efforts and most of them succeeded in completing the task. Values for N were $N=3$, $N=4$ and $N=5$, different children using different values. They had a certain amount of difficulty even in deciding whether a structure they had built was or was not the same as the model. In no case was their solution analytical, and it was clearly impossible for them to reach a mathematical solution (by, for example, extracting the cube root of the total volume). It appeared that many of the children had not developed even a perceptual idea of a cube, and certainly the idea that it had to be as long as it was wide, and as tall as it was long, was very weak with all of them.

This is an example of a stage which is quite clearly not only pre-symbolic, but to some extent pre-perceptual as far as perception of the

cube is concerned. The planes of symmetry of the cube have to be at least implicitly appreciated by familiarity with the details of the appearance of a cube before analytical aspects such as the equality of length, width and height can be realized and then symbolized. The children working in the example above were clearly very far from this stage, and yet they were doing something which was mathematically equivalent to the formula

$$N^3+3N^2+3N+1=(N+1)^3$$

Example 25 Group B Square problem

There were two problems, each involving the construction of a square. In one of them four pieces were provided, one $N\times N\times 1$, two $N\times 1\times 1$, and one $1\times 1\times 1$. In another problem nine pieces were provided, one $N\times N\times 1$, four $N\times 1\times 1$, and four $1\times 1\times 1$. In each case the problem consisted of making a square with all the pieces. The second problem was far easier than the first because all the four lines of symmetry of the square could be preserved internally, whereas in the first problem only one line of symmetry remained internally, and that was a diagonal one. Many children did not consider the latter a square, most likely because of this loss of internal symmetry. To these children a square was symmetrical not only as far as the perimeter was concerned but also as far as what was put inside the square. They could tell a rectangle that was not a square from a true square if they were both plain or regular inside, but of course only by the look of the figures. They were operating on the perceptual plane. It was not difficult, however, to deflect them from this, and they readily agreed that a figure was a square if it was as long as it was wide. This convinced them that both figures were indeed squares.

The children were then asked to rearrange their symmetrical N^2+4N+4 square in such a way that all the units $1\times 1\times 1$ were together in one corner of the large square. All of them found this very difficult, as here again the four lines of symmetry had to be reduced to just one. They experienced similar difficulties with the problem of making a square with one N^2 piece, six N pieces and nine unit cubes. They would make the symmetrical N^2+4N+4 piece, put two more rows on each side of this figure with $1+N+1$ in each, and would then be left with

one unit cube in their hands. The powerful symmetry of the figure they had made deflected most of them from realizing that it was not in fact a square as it was $N+2$ one way and $N+4$ the other way. Again it took some time to persuade them to concentrate on the lengths and widths of the sides rather than on how the interior was constructed.

It is only after discarding the irrelevance of the interior of the square that symbolism of the kind: 'five rows of five' becomes possible. This is followed by a more generalized formulation like 'same number of rows as there are unit cubes in a row', which can eventually be shortened into $N \times N$ or even into N^2, once the idea is well established. We saw in Example 15 that there could well be resistance to such a rapid march towards symbolization if the necessary abstraction is not achieved. If the idea of 'the same number of rows as there are unit cubes in a row' is not a concretization of the 'same number of groups as there are things in a group' idea, then the symbols are not denoting anything very abstract, they are only acting as shorthand for the sentence expressing the idea. To what extent the ability to accept symbolization is tied to the abstraction process is just one of the problems that arises out of this study. The answer is by no means self-evident.

I shall give another example of the incipient symbolism tending to become tied to the perceptual rather than to the conceptual structure. This state of affairs may well be one of the natural stages through which a normal mathematics-learning cycle must pass before it reaches the point of a fully abstract and general realization of a conceptual structure. The perceptual detail helps to build up the mathematical imagery, which may later become stripped of irrelevant detail but strong enough to be used autonomously without having recourse to concrete material. The example below is from Group C, and concerns a stage in the learning of the meaning of fractional exponents.

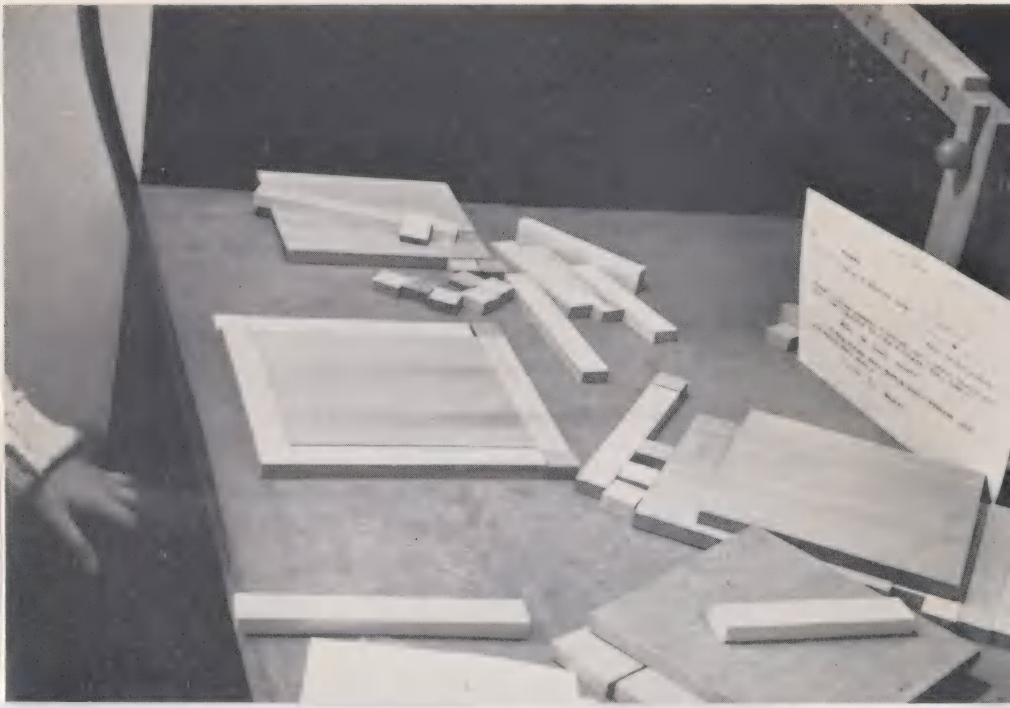
Example 26 Group C Fractional exponents

The interpretation of symbols such as $A^{p/q}$ was still some way off. Although the successions of the operations symbolized by

$$(A^{1/q})^p \text{ or } (A^p)^{1/q}$$

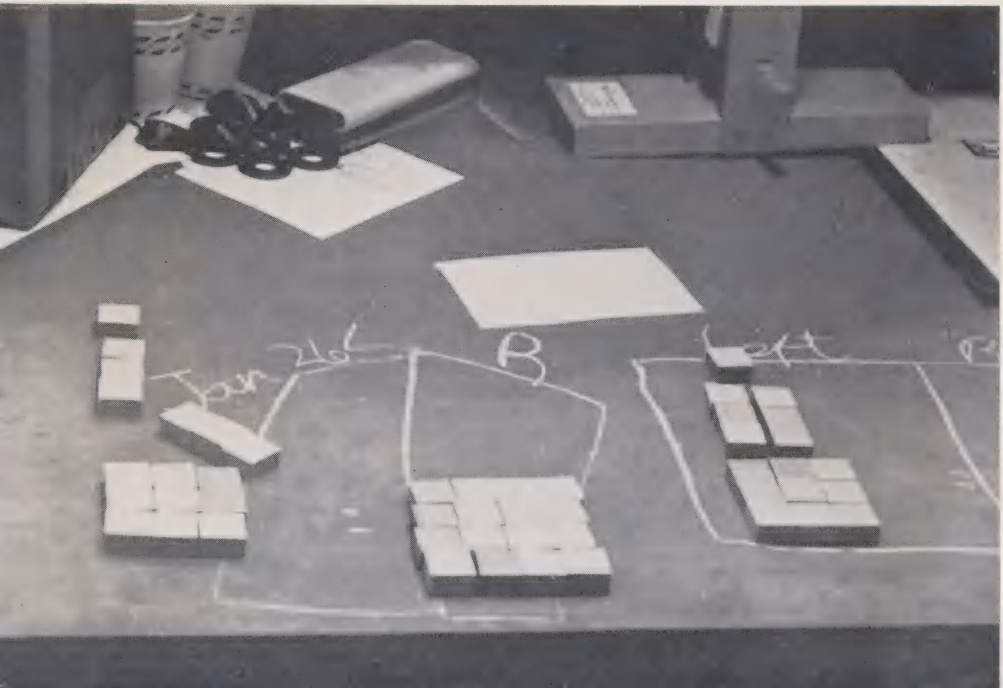
The process of making the square (a) first stage (p. 129)

The process of making the square (b) second stage (p. 129)



One physical equivalent of $x^2 + 2x + 1 = (x + 1)(x + 1)$: $x = 3$

Physical equivalent of transformation of $x(x + 1) + x + 1$ (left) but in transit where $x(x + 1)$ is already split into $x^2 + x$ (p. 137)



had already become meaningful, and even the identity of these two mathematical objects had been hinted at a few times by some of the children in particular cases, no generalization had taken place. When asked what $64^{\frac{2}{3}}$ was, the children unhesitatingly pointed at the piece representing 16 units, and one might have assumed that the symbol was fully meaningful mathematically. However, when they were asked how they might proceed from 64 to 16 by a succession of powers and roots or roots and powers, there was a blank. The association with the $\frac{2}{3}$ power was to the *position* two-thirds of the way up to the sixth power from the zero-th power (the unit cube) along the series of wooden blocks. Two-thirds of the way up to the sixth was the fourth, hence the answer 16 units. Eventually one of the children, and soon afterwards another, discovered that if you took the cube root of 64 you obtained 4, and if you squared 4, you obtained 16, following out the exercise on the series of pieces laid out on the table. On the other hand, they did not associate the cube rooting and the squaring with the figures 3 and 2 in the fraction $\frac{2}{3}$.

The symbolization process, as well as the reverse process of interpretation, is clearly beset with difficulties. Equally clearly, unless both processes, structure to symbol (symbolization) and symbol to structure (interpretation), can be carried out easily, the symbols are not yet doing their job.

In the last example the symbolism was still associated with some perceptual detail of the physical embodiment representing the mathematical structure. The provision of a more varied representation often helps in the process of detaching the symbolism from the perceptual detail. This results in a greater measure of abstraction, as shown in the next example, also taken from Group C.

Example 27 Group C Similarity game

Families to the bases 2, 3, 4 and 5 were again constructed, and Carol very quickly organized the experiences as the second powers covering the members of the family by the next smaller members, and the third powers making the members of each family by the next smaller members. The other children soon caught on as well, recognizing

that the fourth power covered not the next person but the one after that and the sixth power made not the next person but the one after that. It took them some time to find something to do that would use the fifth power. On the 2-family Carol suggested that you could make half the person after the next. Unfortunately this suggestion was not taken up; the best thing would have been to ask Carol if this would work with the 3-family. Instead we suggested that no half-bodies were made and that the only steps allowed in the game were covering and making whole bodies. The children eventually reached the solution, making a number of people of the next size and with these people covering up the person the next size up after that.

Later they built houses with triangular roofs and the same problems were posed. It took the children about a quarter of an hour to realize that the person problem and the houses problem were mathematically really the same. One of the children even suggested that it was a waste of time having so many different shapes, and we might just have cubes, as it was all the same anyhow. The cubes had become symbols, symbolizing third, sixth or ninth powers; the faces of the cubes had come to symbolize second, fourth, sixth or eighth powers, and the bridge to the fully abstract and general concepts of A^2, A^3, A^4, . . . had been built. The concepts were abstract in the sense that a variety of relevant situations in which they were embodied were recognized; they were general in the sense that they could be applied to quite a large class of bases and exponents—even fractional exponents had made their appearance a few days previously.

The table-symbolism for the mathematical group games played by the children in Group E represented for most of them a general as well as an abstract idea. It was general in the sense that they could conceive of a table of any size, and abstract in the sense that they realized that the actual embodiments of the games were irrelevant to the rule-structures of the games. Such a rule-structure was symbolized in a table of moves, which was then more than a shorthand notation for explaining the rules of a certain game; it had become a way of describing a whole class of games that had those rules. The children in Group D did not achieve this degree of abstraction, and so when they made up a table their symbolism did not, for the majority, represent the rule-structure of any kind of game, however embodied. They could conceive of tables

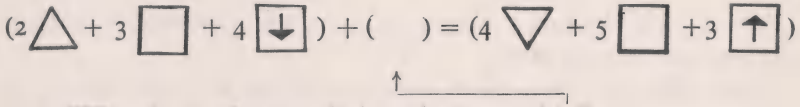
of any size, and in this sense their symbolism was general. It should perhaps be added that they were unable to extend to an infinite game; this might have been due to the limitations of the table-notation. The symbolism can be a limiting factor as well as an aid to thinking.

It is well to remember that a symbol system has two rather distinct uses. One is to circumscribe and fix what has been realized. By establishing a symbol system we can keep before us very much more complex structures than would be possible without such a system. The other function of a symbol system is to be manipulated to find out something which was not known before. One of the intermediate stages in passing from non-symbolic to symbolic thinking in relation to any structure is being able to express the structure learned, and even recognize it and interpret it as meaning the structure which it is supposed to symbolize; yet we may be unable to manipulate the symbolism with sufficient confidence to give us answers to questions. Practically the whole period under review in Group D is a clear example of this. When asked questions about some supposed property of a game the majority of children in this group preferred to go to the embodiments of the games to provide the answers, even after constructing tables from which the answer could have been directly obtained.

At one point it seemed advisable to give these children help appropriate to the stage at which they were working in relation to the appreciation of the symbolism. An aid which was half symbolism, half embodiment was given to help them solve the problems. Here is the example from the protocol:

Example 28 Group D Vector space—complementary addition

D gave them this problem:



D. 'What do you have to do in order to get this?'
J. 'Do you get it?' (This to R)
R. 'Put 4 upside-down triangles.'
J. 'No, put 6, and then take out the opposites.'
L. 'No, it's 5.'

D showed them that the answer for the triangles was 6 upside-down triangles, using what he called his 'fool-proof' scheme, writing beneath the triangles in the given problem the following:

Then the children began to work on the number of cups and squares needed.

It will be seen that J gave the right answer to the problem, even explaining how the answer would in fact fit the problem. The other children, however, appeared unsure. We diagnosed that the symbolism, together with the convention of the opposites being discounted, was not quite established on an operational basis. So the 'fool-proof' scheme was introduced, which broke down the numbers into the groups of corresponding size represented by the numbers, and also gave the children an immediate perceptual peg on which to hang the cancellation of opposites.

Another interesting example of symbolism not functioning for the purpose of enquiry comes from Group C. Here the symbols such as 2×3^5 did express for the children the fact that they were considering two pieces, each containing 3^5 unit cubes. Yet the problem posed to them in the next example was not solved from the symbolism but by reorganizing the pieces embodying the problem.

Example 29 Group C Geometrical progressions

The series to the base 3 was constructed on the table and the following instructions were given:

'Start with the largest piece, take two pieces at a time, and try to build a tower, filling up the "holes" in the tower as much as you can.'

Very soon they had made a representation of $3^9 - 1$, starting with two 3^8 pieces, then two 3^7 pieces, and so on, until two unit cubes were reached. This resulted in almost a cube, but 'there was a little bite taken out of it', the size of one unit cube, as one of the children pointed

out. They then repeated the exercise with one piece each in base 2, three pieces each in base 4 and four pieces each in base 5.

The experience in base 3 was eventually symbolized in

$$2+2\times3+2\times3^2+2\times3^3+2\times3^4+2\times3^5+2\times3^6+2\times3^7+2\times3^8=3^9-1$$

and a corresponding symbolism in the other bases.

We then suggested that only one piece out of base 3 be taken at a time. A tower was built with one piece each right next to the one with two each, but the towers looked too different for the 1 : 2 relationship between them to be apparent to the children. Neither did the formula suggest this relationship. The tower with one piece each was symbolized by the formula

$$1+3+3^2+3^3+3^4+3^5+3^6+3^7+3^8$$

but nobody noticed that this symbolized just half of what the left-hand side of the first formula symbolized. The problem was eventually solved by putting both towers down horizontally on the table, separating the unlike pieces and putting the like pieces together. The 1 : 2 relationship immediately became obvious, and the children were most annoyed with themselves for not seeing it sooner.

We saw in Example 27 an instance of the use of an embodiment as symbolism. This is usually a good bridge between the mathematical structure and its symbolization. The use of the Cuisenaire rods is based on this principle; the rods become for a time another symbolism—to children a much more suitable and convenient one—before they pass on to the more conventional symbols. Embodiments can vary from full representation to very little representation. The MAB are fully representational in the sense that the numerical relationships studied are truly, not only conventionally, embodied in the material. The Cuisenaire rods are also fully representational until the 'crossing over', which no longer embodies but only indicates—symbolizes—the product. The abacus is an example of a partial embodiment; the number of each of the powers is actual, but the powers themselves are conventional. Passing through a series of embodiments with more and more conventionalized properties could be an aid in travelling the road from the pre-symbolic to the symbolic appreciation of structures.

Another interesting example comes from one of the subjects in Group A. This subject was very keen on working symbolically and considered working with blocks somewhat childish. Yet many times it occurred that his mastery of the symbolism was stretched to the limit. In the next example this extreme stage may not have been quite reached, but he certainly had recourse to the embodiment to solve the problem that was put to him. It is not certain, however, whether the picture of what he had constructed suggested to him the transitive property of equalities, or whether the implicit knowledge of this property made him give the response he did, verifying his response by an imaginary transformation of the constructions into one another. Here is the example:

Example 30 Group A Quadratic functions

O. 'With those same blocks, instead of building towers and walls, lie them flat.' (S does so) 'Now remember the other equation you had; would you like to do that with blocks?' (S is not at all keen. However, to oblige, he draws another table like the one he had drawn for the previous problem. He builds it for $F=3$. S has the balance on the table as well)

O. 'Do you know where the equal sign should go?' (S puts in the equal sign between the left- and the right-hand sides of each one of the two embodied equations) 'What else balances?'

D comes back at this point. S indicates that the left-hand side balances the right-hand side of the first equation and that the left-hand side balances the right-hand side of the second equation. Then looking at it again he sees that the squares which are on the right-hand side of each equation are the same thing, so that they are equal. Then looking at it still longer he says:

S. 'The first left-hand side equals the second left-hand side.'

D. 'Well, are they all equal?'

S. 'Yes, every one.'

D. 'How did you know?'

S. 'How I did it is, I took this off, this comes off and goes here.' (Saying this, he reconstructs one of the left-hand groups)

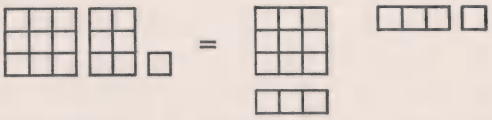
This subject was getting very near to discovering the equivalence group of all the different arrangements into which

$$x^2+2x+1$$

could be transformed. The symbolic expression of these equivalences is not so intuitively obvious as their expression in embodied form. Here for example the identity

$$X(X+2)+1=(X+1)X+X+1$$

is a lot harder to appreciate than the equivalence of the two groups



It is possible to use symbolism sometimes for developing an idea further, provided that the corresponding mathematical imagery is developed. The mental manipulation of the imagery replaces the physical manipulation of an actual concrete situation. Conversely, if the imagery is not well developed, it may be necessary to develop it by concrete experiences isomorphic to the structures being learned before the symbolism can be used as an adequate language by whose means further learning can take place. Here is an example of solving a problem by almost pure imagery-manipulation:

Example 31 Group D Dance-floor game

The rules of the dance-floor game are given out to the children, as well as written on the blackboard (see Appendix I). Then:

D. 'There are 3 girls and 2 men having refreshments, and 7 men, 4 women, 1 boy and 3 girls come to the dance. How many people would be drinking at the bar after they have all taken their partners?'

J. '5 men.'

R. '3 girls won't be dancing.'

J. and R. '5 girls and 5 men.'
D. 'Now 10 men and 5 boys are needed to repair a leak in the roof. How many will be sitting at the bar while the roof is being mended?'
J. '5 women.' (Going back to the first problem)
D. 'That's ancient history.'
J. and L. '5 women and 10 girls.'

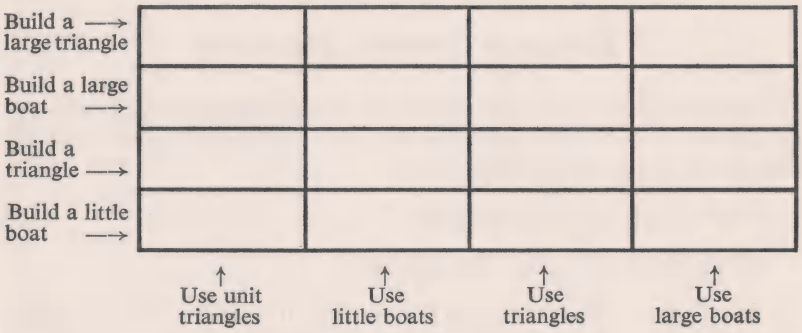
The first problem was quite quickly solved by two children; the second, supposedly harder one, was likewise easily solved by two children. The only aid was the imagery of the dance, the refreshment room, the rules, the leaking roof, etc., which together made it possible to solve some really quite complex problems. The Martian Rockets problem (Example 5) was solved in a similar way, but with the addition of some representative pieces to act as rockets, meteorites and rocket engines; but the imagery of the story must have contributed in some measure to the solution. It is of course difficult to decide whether supplying symbolism to such imagery or imagery-cum-embodiment exercises is not in a sense 'giving the game away' by giving the learner an all too easy channel into which to canalize these experiences. There may be an optimum moment when introducing symbolism promotes insight. Some experimental work on this would be valuable.

We have seen that in a sense embodiments can act as symbols, but in another sense certain types of symbolism can act as embodiments provided the symbolism is close enough to expressing the actual relationship symbolized, or at least a certain aspect of it. Graphs and tables of various kinds come under this category. So do the tables for the games in Groups D and E. There is also an interesting example from Group B, although one wonders to what extent the use of this kind of symbolism was premature with seven-year-old children. The example follows.

Example 32 Group B Calculation of ratios

The children were given the following type of 'matrix', on which the actual figures had to be built. Every cell of the matrix had two arrows pointing to it which contained the instructions for what to build in each particular cell. On one side of the matrix the instructions stated what

was to be built, on the other side they stated what pieces were to be used. This superimposition of the two instructions was very difficult, and



some children had to have explained what to build in each cell. Similar matrices were drawn and given to the children in the bases 4 and 6, and a whole period was spent filling in the appropriate blanks of the matrices. Then another matrix was drawn on the blackboard and the children were invited to fill in the spaces with numbers. The number for a space was to be the number of pieces used in that space. Some of the children found this spatial transfer from the table to the blackboard extremely difficult. They found it very hard to see which were the corresponding spaces on the table and on the blackboard; even when they counted the spaces from one side, they would make a mistake in finding the corresponding row. In fact they found it difficult to establish a simple spatial correspondence between two 4 by 4 squares, one on the table and one on the blackboard. Yet they were required to establish a mathematically much more complex correspondence!

Material used: Triangular-trapezoidal form of MAB.

Bases 3, 4, 6.

(The table is given for base 3)

We have seen examples of imagery used with practically no symbolism; we have also seen imagery used with embodiments as an aid to the imagery (e.g. the Martian Rockets problem). It is possible to use more symbolism to aid the imagery. This was done in Group D, also in E. A neat example of symbolism aiding the imagery can be found in

the Group D protocols on the use of the 'magic' in conjunction with the dance-floor imagery.

Example 33 Group D Magic game

The dance-floor rules were explained to the children, and the people on the dance-floor were permitted to perform the following magic on the people in the refreshment room:

Boy's magic: nothing happens.

Girl's magic: everyone changes sex.

Man's magic: children grow into adults and keep own sex, adults become children of opposite sex.

Woman's magic: children become adults of opposite sex, adults become children of own sex.

D. 'What is the result of using woman's magic on 3 girls and 1 man?'
The children quickly: '3 men and 1 boy.'
D. 'The girl's magic on 3 men and 1 boy?'
Children: '3 women and 1 girl.'

In the meantime, while the children were responding, D was writing on the blackboard:

$$W\ (3\ \text{girls} + 1\ \text{man}) = 3\ \text{men} + 1\ \text{boy}$$

$$G\ (3M + 1B) = 3\ \text{women} + 1\ \text{girl}$$

D. 'What is G (7 boys + 100 women)?'
Children: '7 girls and 100 men.'

The children appeared to have no difficulties with the moves.

D. 'What is M G (7 boys + 100 women)?'
Children (with C in the lead): '7 women and 100 girls.'
D. 'The woman's magic is the same as the girl's magic and then the man's magic.'
J. 'That's like some of the games we've played before!'

D and the children then worked out the following moves:

$$G\,(3B+5M)=3G+5W,$$
$$W\,G\,(3B+5M)=3M+5G,$$
$$M\,(3B+5M)=3M+5G.$$

It should be noted that C was remarkably quick in responding to the transformations taking place.

J. 'The boy's move is like the nothing move, and there are only three real moves. It's like two moves equal a third move.'

Further combinations of moves were then worked out with the children.

Writing the symbolism on the blackboard at the same time as the children were working out the transformations in their heads with the aid of the imagery of the magic story seemed to have a remarkably accelerating effect on the formation of the ideas that were taking place.

8. *The use of symbolism*

So far we have studied cases in which the symbolism, for one reason or another, was not fully operational. Here it was not possible to use the symbolism freely to arrive at solutions of problems without bringing in aids such as embodiments, imagery and the like. The journey does, however, have an end, and one reaches the point when the symbol is so firmly attached to the structure symbolized that it can be freely used in asking and answering questions about the structure.

The boys' reaction at the end in Example 17 (Group B) is a case in point. In both the tree games and the blocks games the boys were able to identify the 'winning throw' just from the symbolic representation of a game. The game had in this case become abstracted from the two games, and the symbolism of a series of numbers such as

(i) 1011010 or (ii) 1010111

referred to something beyond an individual game and therefore quite general, and beyond one type of game and in this sense abstract. It is in situations like this that children can accept the symbolism as well as work with it to answer questions about the structures they have learned.

Example 34 Group A Transformation of problem

The letter Y can stand for any number you like, you can actually choose a number or just leave it as Y.

Suppose you have (Y+4) groups of Y things in each group. How many more things would you have to have before you were sure that you could arrange your things into just as many groups as there are things in each group?

Write down the solution to the problem in the language of algebra.

Try to explain what you have done on the balance.

Try to make up some more like this of your own.

(S starts by copying the equations that were left on his desk from the last time into his book. As he does this he asks, 'Who has the recording machine today?' For the last two days S had been recorded, but apart from having noticed the machine and the microphone on his desk at the beginning of the first period he had never mentioned it)

O. 'Nobody.' (S continues copying the equations. It should be noted that the three equations he has copied are the following)

$$\begin{array}{lll} Y=2 & (Y+4)\times Y+4 & =4^2 \\ Y=1 & (Y+4)\times Y+20 & =5^2 \\ Y=1 & (Y+4)\times 1+4 & =3^2 \end{array}$$

In the second equation, beside the 5^2, he puts $(Y+4)^2$, writing it in the following way without brackets:

$$(Y+4)\times Y+20=5^2 \text{ (or } Y+4^2)$$

O. 'Is it 4 square?'

S. 'No. Y+4 square.'

O. 'Then put the bracket so that it says that.' (He does so) 'Now can you put (Y+something) square in the other two equations?' (In the first one he puts (Y+2) square)

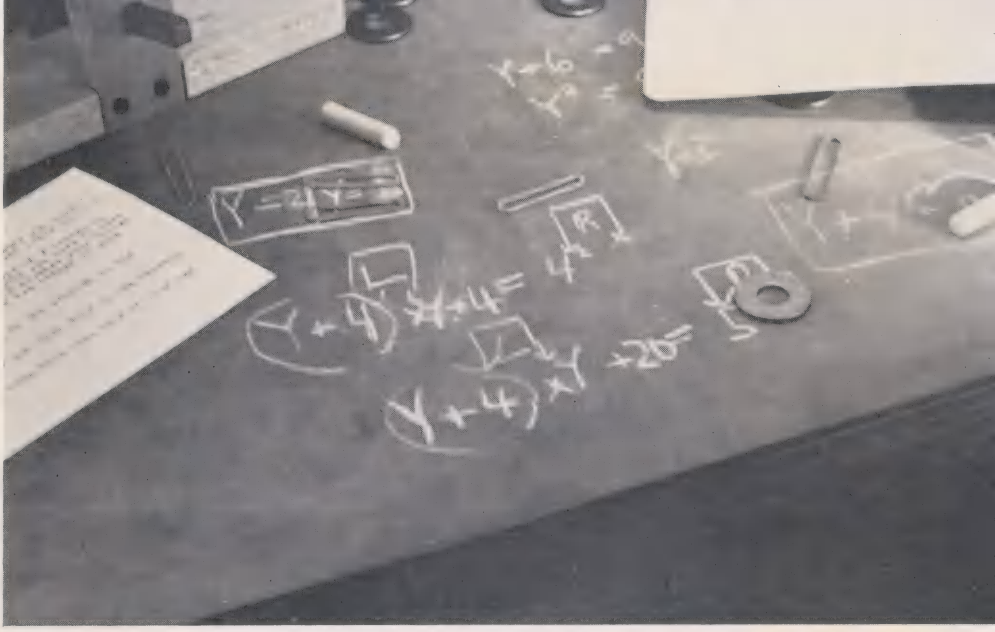
O. 'And in the last?'

S. 'No, it wouldn't be Y+2 square.'

O. 'Where wouldn't it be?'

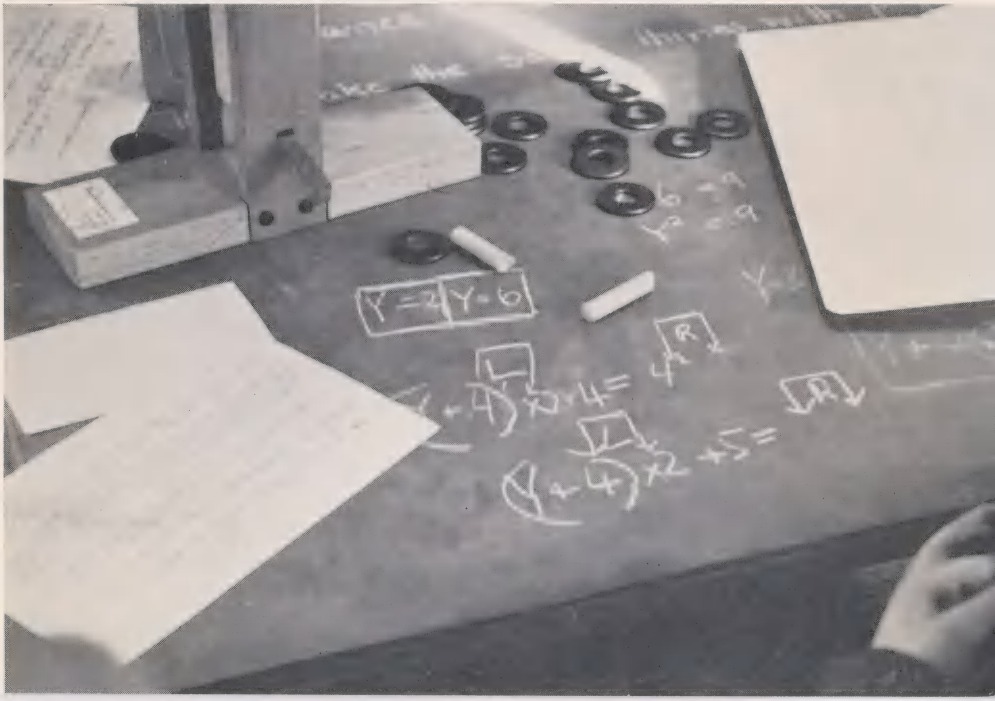
S. 'In the first.'

O. 'Why? What is your Y?'



Symbolic work before transformation of problem (p. 142)

Problem being transformed (p. 145)



S. '2.'

O. 'Then what is $Y+2$?'

S. '4.'

O. 'Then is it right?'

S 'Yeah.' (Then he also writes in, besides the 3^2 of the last equation, $(Y+1)^2$ with the brackets in the appropriate place)

O. 'Good.'

S. 'But the main thing is there' (he indicates the first thing he had written, the $Y+4$ square in the second equation) '$Y+4$ there and $Y+4$ there' (he indicates the presence of $Y+4$ on both sides of the equation)

O. 'Good for you, but there's another main thing to find out. Take another Y.' (S takes Y equal to 3) 'Now add as little as possible to do the problem.' (S writes the following equation)

$$(Y+4)\times Y+0=7^2 \text{ or } Y^2$$

O. 'But does it equal that?'

S. 'Oh.' (He rewrites on the right-hand side of the equation 7^2 or $(Y+4)^2$) 'This could be the same many times.' (Then he picks $Y=2$ again and writes out again the left-hand part of the equation $(Y+4)\times Y+0=$. He does not fill in the right-hand side)

O. 'I think you have to add more than zero. Try it with $Y=3$ again.'

S. (abruptly) 'Wait a minute.' (He goes ahead with his attempt to balance the left-hand part of the equation with $(Y+4)^2$ on the right. He tries $Y=4$ and $Y=5$, each time writing down the left side of the equation)

S. 'U-u-mm. It doesn't work again.'

O. 'Add more than zero.' (S tries again with $Y=2$) 'You've done it with two.'

S. 'No, not the same thing. That's not enough.'

(Note: With $Y=2$, the day before he had done the equation $(Y+4)\times Y+4=4^2$ or $(Y+2)^2$ but this time he has assigned himself a new problem and that is to get the right-hand side of the equation to be $(Y+4)^2$ because there is a $(Y+4)$ on the left-hand side of the equation)

O. 'But you don't want $(Y+4)^2$, you want anything squared.'

S. 'Yes, I do, yes, I do, that's what *I* want.' (Then with Y equal to 2 he writes down the following equation)

$$(Y+4)\times Y+24=6^2 \text{ (or } (Y+4)^2)$$

S. 'I've done it.'

O. 'Good. Can you do it with another Y?'

S. 'Yeah. Any Y. You may need three million, but you can do it.' (Then he starts to take Y equal to 10. At this point D comes up. He hasn't followed S's transformation of the problem, and he sees his last equation)

D. 'Are you sure it's possible for any Y?'

S. 'Yes, sure. I have just to add enough on the left.' (D tries to point out to S the difference between an identity and an equation and to show him that what he (D) wanted S to do was to find an identity, but it is near the end of the period, and S's mind is not open to this new point of view)

The interesting aspect of this activity is that S transformed a problem given to him, without realizing it, and set himself his own problem, which he attacked with great energy and followed through right to its solution, finding that indeed by adding the appropriate constant to the left-hand side of the equation you could always make it equal to $(Y+4)$ squared on the right-hand side; but he was not yet explicitly aware of the fact that this constant depended on the value chosen for the variable Y.

As the observer in the protocol noted, this subject was enough at home with mathematical problems and their symbolization to transform a problem given him into another problem in which he was more interested. He still used the balance, but it is possible that he was merely using it as he considered it to be part of the 'rules of the game'. He was not altogether aware, either, of having transformed the problem. He started with the problem of establishing the identity

$$(Y+4)Y+4=(Y+2)^2$$

where the addendum 4 on the left-hand side was to be determined. He finished up by asking himself what had to be put in the blank in

$$(Y+4)Y+\ldots.=(Y+4)^2$$

What has to be put in that blank to make the two sides equal depends of course on the value of Y chosen, so he transformed a quadratic identity problem into a problem of a function, which turns out to be a linear function. He discovered this later on, when his problem was symbolized for him for the next session. Nevertheless the calculations he had imposed on himself to find the values of this function involved dealing with quadratic functions. It seems that the transformation of the problem from an identity to a function problem involved a kind of playing with the symbolism. It was tempting to pursue this, and we give below his work on the symbolized transformed problem. It must be emphasized that the symbolization of the problem was not done by the subject but by the experimenter.

Example 35 Group A The function problem

Problem I *R's Problem*

Your problem last time was what number to put in the space in

$$(Y+4)\times Y+(\quad)=(Y+4)^2$$

to make it right. We could call the missing number Z. Let us choose a Y, for example $Y=5$. Then

$$9\times 5+Z=9\times 9 \quad \text{so } Z=9\times 4=36$$

Now is it true that $(Y+4)\times Y+36=(Y+4)^2$ whatever Y is? For example could Y have been 3? Try to find out. Or perhaps if you chose Y to be different, then Z would have to be different?

Y	5	1	2	3	4	5	6	7	8
Z	36								

(fill in the Z's)

Problem II

What number do you have to put in the space in $(Y+4)\times Y+(\quad)=(Y+2)^2$ to make it right? Call the missing number Z again, and fill in the table below:

Y	1	2	3	4	5	6	7	8
Z								

(fill in the Z's)

(When he first sees the card S is pleased with the title 'R's Problem'. He sits right down and reads. When he gets to the sixth line he says, '9? Yah-yah. $9\times5+Z$ is 9×9 so $Z=9\times4=36$.' This he is reading. Then he continues. 'U-u-mm. Oh-ho, I see already')

O. 'What do you see?'

S. 'Well, let's make $Y=6$.' (Then he writes on the desk and says at the same time) '$(Y+4)\times Y=60+$, this would be 10, I think, =, no, no, no, no; uh-uh; $+40$' (that is where he had first thought it would be $+10$) 'equals 10^2. No, that wouldn't work because that is 100. Yes, it works!'

O. 'How did you choose 40?'

S. 'It says it in the card. I just proved it.' (This shows that in reading from the card $Z=9\times4$, S has understood that the 9×4 could be written $(Y+4)\times4$)

O. 'Do you want to write that in there?' (indicating the bottom of the card where he could write his results)

S. 'No, I'm going to take another number. You choose.' (O proposes $Y=4$)

O. 'Now, does that work?' (S writes the following equation

$$(Y+4)\times4+32=64$$

and at the same time he says) '8×4 is 32 and 32 is 64. See, that's exactly what he says, and the very problem he said is $Y=4$.' (Now he writes and says) '$Y+4=9$. Now this is going to simplify it down here.' (Farther down on his desk he writes the following)

$$\begin{aligned} &Y=5\\ &(Y+4)\times Y+\\ &\qquad 9\times5+ \end{aligned}$$

(D comes up)

D. 'Are you trying to make sure that what I said is right?'

S. 'I know it's right.' (His attitude is that that is something he has done and has gone beyond. Now he is involved in a more complex task, that is, simplifying the equations. He writes)

$$9\times5+36=9^2$$

S. 'There, it's simplified.'
D. 'Is it true?'
S. (doing the multiplication and addition in a corner of his paper) 'Yup, they do.' (Then he fills in the correct value on the card)
O. 'Try a smaller number.' (S takes $Y=1$. He writes the following)

$$Y=1$$
$$(Y+4)\times Y+20$$

S. 'No, not 20, or is it? Let's see. Yeah.' (Then he writes beside the equation $=5^2$, and beneath the one on the Y-Z value table on the card he enters the 20)
S. '1 we get 20, 2 we get—25, I think. Let's see.' (So he writes on the table the following)

$$Y=2$$
$$(Y+4)\times Y+24=6^2$$

S. 'It works!'
O. 'Did you think it would be 24?'
S. 'No, 25.' (Beneath the 2 on the Y-Z chart he enters 24. Then he considers the value $Y=3$) 'Now let's see. Oh this is easy. 28. That means if you have $Y+4$ it is a times 4, so if it is a $Y+3$ is it a times 3? That's the question!'

Without calculating any more values for Z, S fills in the table.

It is interesting to note that discovering the solution to the problem —that the function required is a linear function whose slope is 4 ('it is a 4 times')—suggested to S that this might be a part of a more general pattern. Here is the true constructive formalist at work, intrigued, as a mature mathematician would be, with the formal structure which is unfolding itself. This shows in a tendency to generalize, to see beyond the particular to a general pattern, and is an instance of the symbolism playing a really creative role. Very likely without the mastery of the properties of the symbolism that this subject had clearly acquired the symbolism could not have played a creative part as it did.

The subject then goes on trying out his hypothesis, but possibly as the observer was not mathematically alert enough to guide his thinking

the hypothesis itself is not properly or explicitly formed. The subject was brought back, with some reluctance, to the problem marked Problem II, whose purpose was to confront him with an identity as a contrast to the function he had just studied. It appears from the protocol that the subject did reach the point of becoming aware of the essential mathematical difference between the two problems. His loud assertion that:

$$(Y+4)\times Y+4 \textit{ is } (Y+2) \text{ squared!}$$

and the dialogue that follows is a fair indication of this. The cognitive strain, however, was beginning to tell at the end of the session, and the escape mechanism of 'going for a drink' came into play when he was requested to make up some more of his own to illustrate the contrast he had just discovered between identities and functions. Remember that this subject was only eight years old; indeed the mathematical feats performed by him were by no means negligible for his age.

Here is the remainder of the account.

Example 36 Group A The identity problem

(S reads)

S. 'He said it already, 4.'

O. 'But try another Y.'

S. '$Y=3$; $7\times3+28$ is 7 squared, 49. I don't get—$Y=3$. . . .' (Here the observer tries to indicate the difference between this problem and the one before, but S once again rejects O's help as he continues to read the card, humming to himself, bom, bom, bom, bom, bom, bom, bom, bom. As O persists in trying to explain the difference to him, S's vocabulary of bom, bom, bom takes on tone variations so that there is a real dialogue between O and S)

O. 'It's different this time.'

S. 'Bom—bom, bom, bom, bom, bom, bom.'

O. 'It's $Y+2$ on the right.'

S. (rather aggressive) 'Bom.'

O. 'You get it?'

S. (with an affirmative nod, and enthusiastically) 'Bom.' (Then he

says) '5 squared is 25, does that sound right?' (At this point there is a great general rumpus over the ruby-headed eraser with all four children involved. It takes several minutes. When S finally gets back to work, he writes down the multiplication involved in the equation with $Y=3$, and checks to see that it balances)

S. 'It doesn't work.'

O. 'Why?'

S. 'Oh, yes, it does.'

O. 'Take another.' (S decides on $Y=1$)

S. '5×1' (and he writes on the table '$5+$')

D. (who comes up just then) '$5+$what?'

S. '5!'

D. 'No.'

S. 'It has to equal 3 squared, . . . then 4! Every time . . . I bet it will be 4, 4, 4.' (S takes $Y=2$ and writes on the desk $Y=2, 6+4=$) 'No, it doesn't work.'

O. 'What's Y?'

S. '2.'

O. 'Then $2+4\times2$.'

S. 'Equals 12, equals 4 squared. So far, correct.' (Note that he made the leap here without mentioning the plus 4, but this was understood)

O. 'Supposing it's 3?' (S writes $(7\times3)+4=25$)

S. '7×3 is $21+4$ is 5 squared. Now, if 6, 7 and 8—I'll do 10 now.' (S writes 10, $-140+4=14^2$) 'Now, let's see, no, it doesn't.' (So S puzzles over this while he produces a babbling noise, but he keeps looking at the card and trying to figure out where he's gone wrong) 'It shouldn't be 14 squared. That was the old way, it should be 12 squared. It works! Now, 5 and 7.' (S writes $Y=5$) '9×5 is 45 plus 4 is 7 squared.' (He writes $45+4=7^2$, at the same time as he talks. Then he takes the chalk and scribbles big check marks all over the table top, with a great deal of show and exhibitionism. Then D comes back)

D. 'Now what is the difference between your problem and this one?'

S. 'I didn't do it yet.' (On the card he fills in 4 for all the Z's corresponding to the Y values from 1 to 10)

D. 'Why is it always 4?' (S simply reads loudly and assertively '$(Y+4)\times Y+4$ *is* $(Y+2)$ squared.' Then D turns to the other side

of the card, to S's original problem) 'If you have a number would it always work on this side?'

S. 'No.' (D turns the card back to the other side)

D. 'And this side does?'

S. 'Yes.'

D. 'Can you make up two more like this?'

S. 'What time is it?'

D. '9.22.'

S. 'It's time, time, time . . . I'm going to get a drink.' (He goes to get a drink and comes back)

O. 'Do you want to continue or are you tired?'

S. 'I guess so, sort of.' (Then he leaves)

It was very tempting to see just how far this subject's power over the symbolism would carry him. Would it carry him over into cubic polynomials without any previous 'embodiment' experience of cubics such as the MAB might give. It seemed unlikely, but it was worth trying.

Example 37 Group A Cubics

Card I Cubics

You know that Y times Y^2 is really Y times Y times Y, or $Y \times Y \times Y$. A short way of writing $Y \times Y \times Y$ is Y^3.

If you have $Y \times (Y^2+2Y+1)+(Y^2+2Y+1)$, can you write down how many Y^3, how many Y^2, how many Y and how many 1's you have altogether? Try.

After some preliminary hedging the subject began:

S. 'If Y was 3, Y cubes wouldn't be . . .' (he interrupts) 'let's take 2, you'd think it would be to the third power instead of the second. You'd think it would be 6, but it isn't, it's 8.' (S reads the problem and starts to work on it. He writes the following in his book

$Y \times (Y^2+2Y+1)+(Y^2+2Y+1) =$ Y^3 Y^2 Y 1

aiming to collect the right number of each power in the appropriate box)

Clearly the subject in this example is trying to do a piece of inductive reasoning which has paid off fairly handsomely in these sessions so far. His strategy is to try a number of different values for Y, and the numbers that can be placed in the boxes and work for all values will be the correct coefficients. The strategy is correct, only the number of possibilities now is so great that success is very unlikely and the subject has this feeling, evidenced by the dialogue at the end of our example. At the succeeding session he goes on making valiant efforts to cope with the same problem, but it defeats him and during the course of the last session he collapses on the floor and has to be bodily carried to his chair. The complexity of the combination of the distributive law and the law of indices (he confuses $Y \times Y^2$ with $(Y \times Y)^2$, although he correctly works out that $Y \times 2 \times Y$ is equal to $2 \times Y^2$) in the purely formal context is just too much for him.

This is probably a case where the power of the symbolism has been heavily abused. It is much like driving a small car at top speed in an attempt to keep up with a big car. No tool is all-powerful; even the most expert mathematician will on occasion be confused by the symbolism of his own invention. Symbolism's power as a tool probably comes from the experience (mental or physical) from which it was originally derived. To step up the supply of power, it is no doubt necessary to go on providing experiences.

One interesting problem is the connexion, if any, between the complexity of a structure and a subject's need to symbolize the structure while or after learning it. A very complex structure needs more symbolizing than a simpler one, so that the learner can keep in mind just what is going on all the time. This was aptly expressed by the children in Group E who said that they needed to symbolize the algebra game they were playing, otherwise they could not remember all the examples they found and then how would they ever find 'the pattern'? Of course the extent of the need to symbolize depends to some extent on the structures present in the culture in which we live. If instead of baseball we had a game whose structure was isomorphic to that of the complex numbers, it would probably be unnecessary to develop a very accurate symbolism for the complex algebra, as the rules

implicit in this algebra would be matters of everyday conversation.

Spontaneous versus imposed symbolization presents an interesting sideline. One fancies that until mathematical conversation finally reaches all strata of society, a certain amount of imposed symbolization will be necessary in mathematics-learning.

Yet another problem concerns the relation between symbols and communication. Symbolism, even language, exists primarily for the personal benefit of the user, first as a recapitulation exercise and then as a tool for developing methods of further enquiry. It is probably incidental that through such symbolism the results of our enquiries can be transmitted to other people, and their results transmitted back to us. The coding and decoding involved in such communications is not at all the same as the *process* of symbolization or its *use* for further enquiries.

The problems of communication naturally had to be faced in a pragmatic way with the subjects in the experimental groups; this was done by devising instruction cards which the children could, we hoped, interpret. Any bottlenecks in interpretation were cleared up where possible verbally.

My hypothesis in *Building Up Mathematics* was that thinking proceeds from the unstructured to the structured, from chaos to order. This is open to question. According to the Gestalt school at any rate a regular imagery, or a tendency to one, is a built-in, innate datum. We tend to see things symmetrically, vertically, etc., even when what we see is not in fact symmetrical or vertical. It is possible that a regular arrrangement is more primary than an irregular one. If indeed this is so it may follow that play passes more smoothly into higher-order cognitive activity if it is directed at the start into regular channels. Such play could lead more easily to an attitude of enquiry about structure, which in turn would free the play-activity. This would give quite a different picture of the sequence of events from what is found envisaged; in many of our experimental situations we found that freeing constraints induced a great deal of self-generated noise. This would hinder higher cognitive activity unless special steps were taken, as with these groups, to reverse the trend.

In fact most children prefer to play games with given rules rather than be told to invent their own game. On the other hand they get a great thrill out of a situation which compels the realization and use of a rule-

structure. It does not seem impossible to put such hypotheses to experimental tests, and we hope that as a result of the preliminary enquiries undertaken and presented here, such experimental work will be taken up.

5

Educational implications

9.1. *Aims of mathematical education*

It is not possible to discuss the educational implications of the work reported here without also discussing briefly the aims of mathematical education, and these are in fact far from clear. The so-called 'practical' people merely state somewhat vaguely that mathematics is necessary in practice, both in everyday life and in scientific work and industry, and so all mathematics ought to be practical—there are no other admissible reasons why children should learn mathematics. Reflection will show that extremely little mathematics is in fact necessary in everyday life. Multiplication is already largely superfluous, and the number of people who are going to need mathematics for their work is relatively small (they could easily be taught it as part of their vocational training). Yet a large proportion of school time, both in the elementary and higher grades, is spent in studying quite detailed mathematics, with admittedly rather indifferent results. It seems that we ought either to reduce the amount of mathematics learnt by children or be rather more clear about why we are making them learn it. Perhaps we have something quite different at the back of our minds when we think about mathematical education—something not entirely practical, namely a feeling that mathematics should add something to the quality of the person who has learned it by allowing him to participate in a cultural stream. But it is quite clear that things have not worked out this way. Mathematics is not used or enjoyed by people after they leave school, so that if our aim was to allow children to participate in this aspect of our culture, we have certainly largely failed. It is legitimate to ask why this is so, and what it is about the present arrangement in mathematics-learning that prevents all but a few children from appreciating the beauty of mathematical structure. There are a number of relevant suggestions we might make on the practice of mathematics-learning as a result of the observations made in these experiments.

Once we have agreed that our aims are not entirely practical, the whole question of what mathematics to teach children is immediately thrown wide open. If mathematics-learning is to be regarded honestly as a cultural activity in the way we regard the appreciation of literature, art and music, then the present syllabuses immediately lose their sanctity. Any syllabus would be suitable provided that with it most children could appreciate mathematics as a beautiful structure, irrespective of possible practical uses. Many parts of mathematics are accessible to quite young children, as our work with all the experimental groups shows. A great deal of the mathematics studied by these children has always been regarded as 'higher' mathematics, and for that reason difficult. Yet there is considerable evidence, even apart from the work reported here, that certain mathematics is often regarded as easier by the lower grades than by the higher grades. The Madison Project, for example, has been giving fourth- and fifth-grade children algebra, including 'difficult' things like quadratic equations, and progress on the same syllabus has almost invariably been more rapid in the lower than in the higher grades. Our groups D and E were able to understand what is substantially the idea of mathematical groups, and to some extent the construction of vector spaces and algebras; Group A made considerable progress with quadratic functions; Group C even finally achieved insight into the overall relationship between powers, logarithms and roots; and Group B were able to achieve quite complex isomorphisms on the place-value concept. All this provides additional evidence that we have seriously underestimated what mathematics children can do and enjoy.

The revision of syllabuses is being undertaken by such bodies as the School Mathematics Study Group, but it would be idle to think that modern mathematics could be introduced into the schools without a considerable revolution in teaching methods. Some pointers about the direction such a revolution might take may be gathered from our study.

9.2. *The dynamics of the learning process*

In *Building Up Mathematics* I tried to outline a learning theory that would account for some of the salient facts about children learning mathematics from their own experience. These observations were carried out in England over several years, but they were not conducted in any systematic manner. The experimental work described in this

publication was based on the knowledge then available of children's thinking processes, which naturally included a great deal of Piaget's work, as well as a series of experiments I had conducted in Leicester and reported in an earlier publication.[1] This learning theory was to be based on four principles. The first was the *dynamic* principle, which in its primitive form merely stated that learning proceeds from experience to the act of categorizing. In its more detailed form it stated that learning proceeds in cycles which follow one another in regular succession. Each cycle consists of approximately three stages: a preliminary and rather unstructured 'play' stage, a more structured intermediate constructive stage, followed by the insight, then an anchoring stage in which the new insight is more firmly fixed in place. The second principle was the *constructivity* principle, which stated that construction always needs to precede analysis; with children there is usually a preponderance of constructive over analytical thinking, analysis appearing rather slowly and gradually as they grow up.

As a result of the work with the experimental groups described here, it is necessary to modify these two principles. It is no longer quite clear that cognitive organization proceeds from the unstructured to the structured, from play to construction. Nor is it clear whether in the beginning there was chaos, as the Greeks thought, or the Word, as the Hebrews thought. Naturally there is a great deal of chaos in the beginning of everything, but there is also at least a tendency to order, and possibly, if the Gestaltists are to be believed, to order according to certain principles. So even if a situation seems somewhat chaotic we have to perceive it as potentially more orderly than it really is if we are to evaluate and use it. And if this is so, then a certain primitive construction must precede 'play', or what would there be to play with? In the play-diagram drawn on page 25 this is expressed by the arrow passing from manipulative play to construction, where the most primitive manipulative play is the first tentative groping towards construction. The constructions are then re-examined and manipulated again, either directly or through analysing the rule-structure. This leads back to manipulative play on the next level and another cycle is started. It is of course difficult to see how one can construct something if there is nothing out of which to construct it, just as it is difficult to see how it is possible to play without first having constructed something to play

[1] *Concept Formation and Personality*, Leicester University Press (1959).

with. It is the old problem of the hen and the egg. Fortunately for educational practice, if not for theory, by the time conscious education of children becomes possible, both the hen and the egg are well established and the problem remains academic. Perhaps we can formulate the process in the following way.

The unstructured play, which is assumed fundamental in the primitive theory, can be described in our terminology as manipulative play. This leads to the construction of more and more complex categories as the cycles progress; playing with each construction follows, and enables the player to discover the regularities. These regularities—the rule-structure—once discovered are subjected to either retroactive or progressive analysis; this is the anchoring stage of the cycle and helps to fix the properties, possibilities and limits of the rule-structure in theory and in practice. But the rule-structure is not always consciously analysed, particularly not by young children; often it is stamped in and made acceptable and operational by repeated use and practice in recognizing situations where the rule-structure is applicable. This is the stage referred to in the primitive theory as the *practice stage*, which leads to possibilities of manipulation on a higher level.

Both the constructivity and the dynamic principle are incorporated in the model just described, of which a spatial representation is given in the first chapter. Much experimental work is needed before we can be sure of the structure of the learning process, and then educational experiments will be necessary to establish how learning abstract structures may best be fitted into particular culture patterns. The effective use of manipulatory, representational and rule-bound play must be investigated, after the functions of these activities have been established and described in theoretical models. There are a great many problems connected with the development of analytical powers in children, particularly whether the growth of such powers can or cannot be accelerated effectively by suitable training.

The educational lesson to be learned from our experiments is surely that every part of the learning cycle deserves as good a chance as possible of coming to fruition. Children should have structured materials to experiment with, to allow manipulative play to do its job. And they should be allowed to make their own constructions from these manipulations; a construction forced by the teacher is likely to result in parrot-like repetition of responses and is not likely to be

genuine at all. The role of the teacher should be to guide, and to provide an opportunity for making discoveries, rather than to teach directly. Group discussion is a good method both of introducing a new idea and of summing it up, but in between these two the creative and constructive work must be done by the child actively to enable him to reach an end-point which is both intelligible and satisfying. In all the groups except Group A there was a certain amount of group discussion between myself and the subjects, both at the beginning and at the end of a cycle. The summing up, particularly in Group E, often resulted in new hypotheses springing up 'from the floor', and tended to start other cycles with some of the subjects.

A problem that urgently needs tackling is how group-work with part or all the class can supplement individual learning. Since children appear to proceed in such different ways and at such different rates, individual work of some kind seems inevitable. On the other hand, discussion in small groups is known to be very stimulating to thinking, and it would be foolish not to make use of it for the acquisition of abstract structures. Experiments are now in progress in Leicestershire, and in Adelaide, Australia, which should help us to assess the results of mixtures of group and individual work.

9.3. *Abstraction and generalization*

We have seen how the first two of the four principles can be modified and extended. These were the dynamic and the constructivity principles, which can now be replaced by the extended dynamics principle just described, which incorporates the constructivity principle as part of the description of the cycles. There were two other principles: the *perceptual variability* principle, and the *mathematical variability* principle.

The perceptual variability principle stated that to abstract a mathematical structure effectively, one must meet it in a number of different situations to perceive its purely structural properties. The mathematical variability principle stated that as every mathematical concept involved essential variables, *all* these mathematical variables need to be varied if the full generality of the mathematical concept is to be achieved. The application of the perceptual variability principle ensures efficient abstraction; the application of the mathematical variability principle ensures efficient generalization. By abstraction I mean understanding the structure's applicable breadth; and by generalization, grasping the

full extent of the mathematical class, usually of numbers, for which the expression of the rule-structure is valid. Effective mathematical thinking must take into account both the abstraction and the generalization process.

We used the perceptual variability principle in all the experimental groups in this study by making it a practice to present mathematical structures to the subjects in multiple embodiments. In Group A, however, a considerable amount of work was done in one embodiment before proceeding to the next. The subjects at first worked exclusively with wooden squares and cubes, and then exclusively with the balance beam. It was only after these 'sets' that they were subjected to any simultaneous 'bombardment' by several embodiments. In the work with the other groups there was an immediate mixture of embodiments. It may not be coincidence that perceptual blocks were only encountered in Group A, as these may have been the result of working in one embodiment for too long. The subjects attached importance to features of the embodiment which were quite irrelevant to the mathematical structures they represented. This happened with the geometry of the square in the case of one subject, and stood in the way of her developing the mathematical concept of a square. Variation of the embodiments minimizes the chances of such a block occurring, but another danger, the extraction of a pictorial communality rather than of a common essence, arises.

Younger children, as Piaget pointed out some years ago, tend to think in terms of the perceptual detail rather than of some invariant structure, and if they are young enough, the treatment by multiple embodiments is almost certain to yield the extraction of pictorial communality rather than the generation of an abstraction. Here again, it remains to be seen whether abstraction ability improves with the practice of the abstraction process. The work with Group B indicates that even young children (aged 7) can be induced to abstract through the use of multiple embodiments, although they find this rather difficult. The developmental problem in connexion with the abstraction process has already been mentioned, and we hope that research will follow on this.

There are a number of educational problems connected with the application of the mathematical variability principle. We have many examples of subjects able to generalize two variables but not three.

To what extent is artificial restriction of the generality of a structure desirable in the interests of learning? Is it possible to present a general structure for perhaps a longer period, at the end of which the generalization is made at once to the most general case, or is it sometimes wiser to use discretion as the better part of valour, and increase generality bit by bit? The work with the MAB acts on the first assumption, and a number of the instructions for the children in the use of the algebra experience materials (used in Group A) took the line of generalizing one by one. There may be individual differences here which are worth investigating.

The fundamental problem is whether generalization should take place simultaneously on a broad front, or on several narrow fronts followed by abstraction into a broader front later. Younger children seem to find generalizations on a narrow front, i.e. within certain well-defined fields, easier. This may be part of the developmental pattern.

9.4. *Symbolism*

One of the problems about using symbolism is how to find the best time for introducing it. If this is done too early, it tends to be an empty shell. Classroom work in mathematics can so easily degenerate into learning certain rules by which these signs can be manipulated, and studying situations in which they are applicable, each application being separately learned. This of course is necessary if the signs do not symbolize anything. On the other hand, it is possible to wait too long before introducing symbolism. When a child has become familiar with a mathematical structure he needs a language in which to talk about it, think about it, and eventually transform it. New constructions need new names, their properties must be described by new symbols if more of the detail of the structure is to be grasped at one time, and so reflected upon more effectively. In Groups D and E, but particularly in Group E, the introduction of the matrix notation to express the rule-structure of a game gave the children who were able to use it a very powerful weapon for attacking the problems connected with these structures. It may also have helped them to round off the abstraction process by making them aware of exactly what it was that two seemingly different games had in common.

There is also the question of whether symbolism can be used as a tool for cutting through irrelevant noise during the abstraction process,

or whether it can only be used to formulate what has already been abstracted. If the latter, then every learning situation will have a ceiling determined by (1) the amount of noise generated, (2) the amount of noise the learner is able to cut through. As we have seen in the discussions on play, free play generates more noise than restricted play, but it also generates more zest, although there is also a measure of excitement in the restrictions and rules imposed, especially if they are naturally imposed or self-imposed. If a situation generates more noise than the learner is able to cut through, learning the relevant abstraction becomes impossible. Cutting down the noise by imposing restrictions *may* also cut down the supply of energy. The problem, then, is essentially that of finding the laws of interaction between the *three* variables:

(1) Amount of noise generated,

(2) Amount of noise learner is able to cut through,

(3) Extent of motivation of the learner.

These variables are in turn dependent on others which it may be possible to manipulate to improve the learning situation. For example, the amount of noise generated can be controlled by the number and the kinds of embodiments provided. In fact, the amount of noise the learner is able to cut through can possibly be increased by practice in selecting relevant information from noisy situations.

To return to the original problem, one way of introducing symbolism is to provide half-way houses between the experiences and the symbols representing the experiences. For example, if 'vectors' are introduced which consist of squares and triangles, a sequence of notations such as the following might serve as a bridge between the experience and the symbols:

Here and are opposites and and are opposites,

and the following 'dictionary of symbols' is used in the third stage:

s $\bar{s}$ t $\bar{t}$

A further refined stage might be

$$(3, \bar{2}) \text{ and } (\bar{2}, 4)$$

leading easily to the conventional notation of

$$(3, -2) \text{ and } (-2, 4)$$

for two-dimensional vectors, and so on to vectors of any number of dimensions through the introduction of more objects in 'opposite' positions.

This maps a symbolic path which moves gradually from close correspondence with what is symbolized to conventional symbolism. I have already suggested that a series of embodiments becoming more and more conventionalized may also be used for the same purpose, leading the learner to a position where an appropriate symbolism may be more readily accepted.

We have seen how important constant feed-back into experience is. It is extremely doubtful, for example, whether the subject quoted in Example 30, although he appears to be quite at home with the symbolism, would have arrived at the conclusions of transitivity without the possibility of feed-back into the patterns which acted as an anchor for the symbols. The subject even explains how he did it: '. . . I took this off, this comes off and goes here.' The transformation was clearly not done with the symbols, but through feed-back to the experiences from which the symbolism had been originally derived. It is very easy for symbolism to assume an autonomous role if feed-back is not applied from time to time. No doubt this is what happens in most mathematics lessons for most students; they learn to handle the symbolism autonomously, and so the structure they are handling ceases to be symbolism and is divorced from all reality. Perhaps this process has gone to such lengths because of the formalistic climate of opinion that prevails among mathematicians. A trained mathematician does not need feed-back; he can quite safely play with the symbolism creatively. This is not true, however, of the majority of our mathematics students today.

Example 34 is a rare example of the symbolism taking over as a creative agent. The subject plays with the symbolism to the extent of transforming one problem into another different problem. The same subject, however, is quite over-stretched in Example 37, where cubics are introduced by extending the symbolism without any possibility of feed-back, and his failure (he was a highly intelligent subject) to cope with this situation can in all probability be attributed to this. The symbols had grown, or rather had been made to grow, out of previous symbols, and their 'meaning' became more and more vague. The 'meaning' of a symbol, psychologically, is the imagery evoked by it; mathematically, it is the place the concept symbolized has in the wider

structure. It is safe to say that most children do not attribute any psychological meaning to symbols like x^2, x^3, etc.; such symbols do not evoke any corresponding imagery—or if they do, the imagery is likely to be geometrical (say of geometrical squares and cubes) and not corresponding to an abstraction.

The process of connecting symbolism to imagery is at the heart of mathematics-learning. It can be done by means of 'cover stories', or through embodiments. We have seen that the introduction of symbolism can promote the development of imagery, as in Example 33. And, of course, conversely the introduction of imagery can make a system of symbols acceptable, as in the magic game or in the mouse-hamster-guinea-pig farm game. The use of multiple embodiments also has the same purpose—to enrich the imagery which should eventually be evoked by the corresponding mathematical symbolism. The poorer his imagery, the more embodiments a student is going to require to be at home with the corresponding symbolism.

Perhaps a certain distinction will have to be made between work-symbols used as a part of the mother-tongue, and mathematical symbols developed to denote actual mathematical operations and relationships. The *name* for an operation serves a somewhat different purpose from the *symbol* for that operation. The plus sign is not the same as the word addition. The word addition describes what it is that we are thinking of when we add; the plus sign, meaning the operation of actually adding a number to another number, symbolizes this act quite precisely. The plus sign is a predicate which is defined on three subjects. For example, we might say P(3, 4, 7) is true as $3+4=7$, but that P(3, 4, 6) is correspondingly not true. The predicate P(x, y, z) is true if $x+y=z$, and not true if $x+y\neq z$. This predicate P is a part of the system of mathematics expressing relationships between numbers by means of operations. The word addition is a word we use to describe a certain aspect of this system, and as such is in a meta-system, in a meta-language. It is just as important to be able to talk about a language as to talk the language itself, and from the educational point of view it is important that both the meta-language and the language be learned. There is a certain cultural difference here between Great Britain and the United States; in Great Britain less importance is attached to the meta-language, whereas in the United States the use of the right terms is considered very important.

The mechanical use of symbolism, much as a computer would be used to save labour, presents yet another problem. Mathematical thinking, like any other kind of creative thinking, has no predetermined end and cannot be completely mechanized. This does not mean that certain phases cannot be standardized to become quite mechanical. This is what happens when, for example, we use a formula and substitute particular values for the variables, or when we transform a formula into another one according to some definite rules of manipulation. It is quite unnecessary to keep in mind all the time exactly what is happening from the mathematical point of view while such substitutions or transformations are taking place, as long as the possibility of interpretation is constantly there, should a non-mechanical step in the form of a decision be required at any stage. The situation could perhaps be likened to the motor skill of walking up a hill; we do not need to be aware exactly what is happening to our limbs and where our feet are being placed, provided the road is fairly smooth. If the road ceases to be a 'standard' sort of road, the mechanical method ceases to be applicable. If we are climbing up a mountain a rock-climb may be necessary as part of the ascent; and then our awareness of the detailed activity must be considerably higher than during a walk up the 'standard' road. Just as we must be aware of every foothold during our rock-climb, so we should not shrink from employing an even greater degree of mechanization on standard stretches, such as taking a train, bus or plane. This is perhaps the best attitude to adopt to the introduction of teaching machines. There are bound to be people so dazzled by the possibilities opened up by further mechanization that they will believe every part of the way is equally susceptible to mechanization. This should not prevent us from developing a more sober view of mechanization; it is bound to be appropriate in certain circumstances and not in others (another fruitful field for educational research). The present trend towards programmed textbooks and teaching machines merely attempts to perpetuate the current rather imperfectly mechanized learning by improving the techniques of such mechanization. The fault is thought to lie in the inefficiency of the mechanization, and not in the lack of opportunity for the learner to think independently. Although it cannot be denied that subjects can be conditioned into giving certain 'right' responses to quite complex sets of interdependent stimuli, I suggest there is more than this in learning mathematics. The exposure to a

truly mathematical, open-ended situation is possibly a rather important ingredient of mathematics-learning. If the best way of learning something is by doing it, this surely applies to mathematics; we learn by actually doing, constructing, mathematics, rather than by going through predetermined sequences where each act prompts the next in a mechanically determined way.

9.5. *Some educational problems*

This problem of programming is present in a modified form in all types of instruction where a certain curriculum, however vaguely defined, must be covered within a certain time. To what extent can we afford to 'waste time', as some teachers put it, allowing children to fumble about trying to find their own way when it would be so simple just to tell them what to do? In other words, how much should be given away and how much left for the children to find out? To answer, or even to begin to answer this question, we must hold before us our avowed aims of mathematics-teaching. If the aim is the acquisition of techniques because these are going to be useful, then we are tempted to think that time allowed for thinking about the principles involved is wasted, although even this is by no means certain.[1] If the aim is to add something to the learner's personality by the integration of a certain live piece of mathematical experience, then we are less likely to assume that thinking time is time wasted.

The time element in turn raises the usual question of whether to 'push' children along, and, if so, to what extent, and how. Children pushed beyond their capacity, either intellectually or emotionally, will resist in some way. The children in Group A were subjected to considerable pressure—the very fact that they were outnumbered by the adults in the room must have contributed to this. All of them resorted to defence mechanisms of one kind or another during the sessions; this they were able to do without being aggressive because of the permissive atmosphere created for them. They resorted freely to going for drinks of water, periodic projective play, and a number of other devices when the going got too tough for them, and this provided a 'safety valve'. The usual scholastic pressure applied to children to induce them to learn takes the form of a punishment-barrier which checks all behaviour except the learning intended, although in fact it is never quite

[1] See Katona's experiments on transfer.

possible to achieve this as children can usually find some ways of circumventing the punishment-barrier—by deception, for example.

The children in Group D also felt under considerable pressure, if only because they were at a more conventional school than those in Group A and they would naturally regard school as a place where they were under pressure to perform. Defence mechanisms are not normally allowed to function in the conventional school (if they were, very little learning would take place), and the mounting pressure found expression there in more aggressive behaviour towards the adults, particularly on the part of the boys. In Group C, where 'getting on' was also in the air, the children would sometimes resort to play as defence, in the same way as the children in Group A. The only group where practically no defensive behaviour was noticed was Group E. There was less pressure there, as no particular goal was set by the experimenters, and a choice of four paths was given each child: (1) group theoretical games, (2) vector-space and algebra games, (3) transformation of vector spaces, and (4) restrictions in vector spaces (sub-spaces). The children chose their own paths and worked along them very assiduously. One child invented what he called a 'machine' for working out triple products in a six-element group by rotating a number of superimposed hexagons; others worked on the structural properties of the farm-game or the dance-game; others were transforming vector spaces, or finding the multiplication tables of groups embodied in games. This created a working atmosphere of enquiry from which no escape seemed desirable, and so defence was unnecessary. It is important to ask what conditions will permit this kind of highly motivated learning which keeps up its own impetus. Possibly the variety of representations, here in the form of stories, embodiments and symbols, as well as the choice of mathematical paths, contributed to the success of this group.

The use of contrast presents a further problem. Very little was used in the experimental groups, and it was only when the work was well under way that we realized some of the difficulties the children were experiencing might be due to the lack of contrast situations which would have enabled them to see the relevant relationships in a clearer light. Most of the group theoretical games were 'good games' and so were exemplars and not counter-exemplars of mathematical groups. Group A learnt commutativity, but it was in Groups D and E that non-commutative situations were encountered in the form of non-Abelian

(non-commutative) groups. It is quite easy to pinpoint the commutative property of operations by pointing out such non-commutative acts as opening the window and putting your head outside. Non-distributive situations are also easily found; for example, cooking is not distributive with respect to mixing. Cooking a mixture of eggs, milk and flour will produce a pancake, but quite a different result is obtained by cooking all three separately and then mixing them! A non-associative situation is obtained by people marrying and having children. Take, for example, three people: John, Mary and Michael. John and Mary marry and have a daughter, who marries Michael, and they have a son. This would clearly not be the same child if John married the daughter born to Mary and Michael, were they to marry, and John and this daughter have a son.

There are also a great many problems connected with the dynamics of the learning cycles. The model outlined in the first chapter should be tested further to see whether it really is a feasible way of accounting for the facts, and if necessary it should be modified and developed. The working of the parts of the cycles should also be experimentally investigated: the passage from manipulative play to construction, for example, would repay detailed study. One such study was done over a period of one year in a first-grade class in Florence, Italy, and an abstract is given in Appendix II. There appear to be certain fairly well-defined stages through which children pass when presented with the MAB and allowed to do what they like with them. Whether these stages are general, or whether they are influenced by the culture patterns in which the children grow up, could form an interesting anthropological enquiry.

Probably some people tend to abstract more readily than others. These people think more readily in terms of classes than of individual situations. To a quick abstracter a situation is already an exemplar of a lot of other situations of similar structure. To a slow abstracter this will not be apparent until he has met a number of situations exemplifying the same structure. If this is indeed so, then the pedagogic problems of the quick and of the slow abstracter are rather different. The quick abstracter will have extracted an abstraction from a small number of situations of one type; to give him a learning task embodying the same structure in a different situation may confuse him, as he may look for some complication in the 'noise' of the second type of situation instead

of realizing that the second type of situation is simply another embodiment of the structure he has already abstracted. He may see more in the task than there is in it, and the clarity gained from the first situation may be lost. The slow abstracter on the other hand must be helped by providing more exemplars which he can consider simultaneously. Since a common essence can only be extracted from a number of situations if they are somehow simultaneously scanned, the problem of effecting an abstraction process is a particular case of the problem of keeping more than one thing in mind at the same time. Every teacher of young children knows how difficult this is to achieve, and how children are better and better able to concentrate on more than one thing as they grow older. It would be interesting to know whether this ability will improve by practice or whether it is largely a matter of developing maturity.

Readiness to abstract is very probably intimately connected with the richness of a person's imagery and its ready availability. There must be a good store of images to select from: the good abstracter will know how to select the relevant images which 'go' with the experience he is having, and so make a satisfactory classification of his experience. One hopes that the use of multiple embodiments helps to enrich the imagery of the slow abstracter.

No doubt the solution of some of these problems will give rise to fresh problems which cannot as yet be formulated. Research is fortunately in progress all over the world into the many aspects of these problems and I shall conclude this chapter by giving a short account of the present situation.

9.6. *The present state of research*

There are three main areas in which research into the problems of teaching and learning mathematics is now in progress. Bearing in mind that there is no hard and fast boundary between these, they are:

(a) curriculum research,
(b) research into methods of teaching,
(c) research into the cognitive processes which permit mathematics-learning to take place.

The questions surrounding the optimum mathematical curriculum can roughly be divided into two kinds of questions:

(1) What could be taught? and

(2) What should be taught?

Work done in the Madison Project, the School Mathematics Study Group, the Leicestershire Mathematics Project, Jaskowski's project in Poland, and so on, all show that we have certainly not reached the limits of the mathematics that could be taught in schools at quite a young age. Research into the possibilities of extending the curriculum is bound to be tied up with method-research; for example the calculus would in all probability have to be presented differently to twelve-year-olds and eighteen-year-olds. In the selection of methods we must be guided by what we know (however little) about the fundamental thought processes that enable mathematical thinking to take place at all. So the work reported below must be considered in a sense as related team work. Indeed, formal co-ordination of the work in progress is long overdue, and efforts are now being made to establish the skeleton of an international organization which could be used for such co-ordination and planning.[1]

The work of Papy in the field of curriculum research is interesting. It confirms, for example, that the mathematical theory of groups can be taught in the form of class lessons to quite young children, as we found in the brief run of sessions in Groups D and E. In Poland, the work being carried on in Cracow and elsewhere points in the same direction. Some form of experiential background is given in all these studies to initiate the growth of the concepts.

The Madison Project, based on Webster College, St. Louis, Missouri, is concentrating on work from fourth to seventh grade, adapting the use of frames from equations to identities. Children discover axiom systems by gradually shortening the list of true statements about variables and their connexions until they are reasonably certain that no statement on the list can be deduced from the others on the list and that the list cannot be further shortened. Quadratic equations are introduced by playing with the connexion between numbers that fit into the frames to make the statements true and their coefficients. The approach appears to depend on the assumption that the symbolism used is perfectly meaningful and operational to the children, so much so that they can play with it. The motivation comes from the excitement

[1] International Study Group for Mathematics-Learning, 200 California Avenue, Palo Alto, California, with centres in U.S.A., U.K., Poland, Italy, Hungary and Australia up till now.

of manipulating a complex structure and finding ways about it and through it. Children certainly appear to enjoy the lessons and there is much to be said for investigating the practicability of such methods in other parts of mathematics, as well as for investigating in detail the thought processes evoked by such symbolic play.

The School Mathematics Study Group is making an important contribution in the United States to curriculum research. They have reached down as far as kindergarten level, and the lower the curriculum investigations go the more difficult it becomes not to raise questions about the methods of imparting the proposed curricula. Inductive approaches are naturally desirable at the early stages, but in many cases the work proposed is much too formal, and it is not very likely to have much transfer value, even if it is learned by young children as they suggest. Here again we see the need for getting together with another group whose task would be to carry out an exhaustive study of methods that would work in practice in the lower grades and make the excitement of modern mathematics accessible not only to the bright but to the 'average' or even so-called 'dull' student.

There are a large number of other curriculum projects working in the United States at this moment. The principal one is probably the Illinois Project, where the 'arithmetic with frames' was started. These projects will probably be useful as long as they are able to keep their feet on the ground (by being in constant touch with actual situations in the average classroom) and as long as they are able to enlist the *active* co-operation of the schools. The Illinois Project has been notably successful in doing these things, and much valuable information may be expected from it in the near future. The Minnesota School Mathematics Center, led by Paul Rosenbloom, has started a very interesting project, appealing to the children's imaginative ability. And Pat Suppes, of Stanford University, has also worked on a new 'sets' approach, promising interesting results.

In mathematics projects in Leicestershire, Hawaii, Adelaide and elsewhere, it has been established quite conclusively that children of seven and over in an all-ability class can cope with the abstractions and generalizations required for learning the concept of number systems, including carrying out all arithmetical operations in any arbitrary number system. It is certain that children of between nine and eleven can learn the elements of identities and equations in algebra, linear and

quadratic alike; they can handle linear equations in many varied situations. Eleven- and twelve-year-olds can learn powers, roots and logarithms, including the summing of geometrical progressions, fractional exponents and so on. Experiments are now proceeding with an eleven- to twelve-year-old group who are beginning their algebra course with directed numbers, including complex numbers and other multidimensional structures. We are still very far from discovering the limits, if indeed there are any definite limits.

Research into methods of teaching has been rather fragmentary. There has been no generally agreed theory of learning that would have enabled experimenters to make plausible hypotheses about which methods would be more successful. A comprehensive learning theory which would account for the learning situations found in classrooms does not of course exist, much less any theory that would predict what might be possible under previously determined conditions. Such a theory will have to grow eventually out of the concerted efforts of practical experimenters, fused together perhaps by a body of theoretical workers aware of the experimenters' successes and failures. This report is an attempt to set up a theoretical model which we hope will have predictive value. If its predictive value is still too low to be a useful guide, another model or a different interpretation of this model will have to be thought of, to move nearer to an 'explanation' of present processes and effective prediction of future ones.

There have been a number of suggestions lately for changing teaching methods in the mathematical classroom, and almost all are worth consideration. It should be possible now to learn both from our successes and failures; indeed the lessons of any failures are very often more important than the successes.

Gestalt theories prompted Catherine Stern to initiate her method of the 'structural' approach to arithmetic-learning. The general idea of teaching structure rather than responses to sets of cues is of course one that would commend itself to most mathematicians as well as to quite a lot of psychologists. There are probably two major shortcomings in the ways in which this method is applied. One is that arithmetic may be regarded as distinct from the rest of mathematics and so does not prepare children for a broader understanding of the whole subject. The other shortcoming is one that any *single* embodiment of mathematical ideas possesses; this is that there is always the danger of the kind of

perceptual blockage we have seen occur in Group A after the exclusive use of one material for a certain length of time. In other words, too strong an association of mathematical manipulations with one kind of material may have the effect of making other applications of the mathematical structure more difficult; sticking to one material might interfere with readiness to transfer.

The Cuisenaire material, consisting of a set of coloured rods, each colour corresponding to a number between 1 and 10, suffers from the same defect. It is, however, less restricted to arithmetic than the Stern material, as the lengths of the rods are not obviously divided into unit lengths and so lend themselves very well to the study of ratio and then fractions (a reversal of the conventional order), and also to the formulation and solution of linear equations and a number of other mathematical ideas. The coloured rods will probably acquire a permanent place in the educational armoury of mathematics-teachers, particularly if it becomes the practice to use them in conjunction with other materials or in situations whose mathematical structure is identical with the mathematics being learned through the rods.

Both these materials have 'work-books' to go with them. These can be dangerous things to have with any structured material as using them may encourage a return to the old system of learning how to respond to certain cues. Ideally the use of structured material is designed to enable children to discover for themselves the mathematical structures we want them to learn. If the path is too rigidly laid down, this discovery can easily become a mockery of what was intended. If, on the other hand, the situation is left too open, purposeful learning may not take place at all. Our problem with structured material lies in determining just how much it is useful or wise to give away, and in what way and how much we can leave for the child to discover. This problem is still very far from solved, and it will take time, effort and a great deal of ingenuity to get down to a detailed and practical solution the classroom teacher can use in everyday situations.

There has been an enormous volume of research into details in the teaching of arithmetic; whether equal additions or decomposition is the more effective method of teaching subtraction, for example. These have invariably ignored an important factor—the child's understanding of the process being learned—because up till now no effective group tests have been designed to measure the extent of a subject's under-

standing of a concept. For this reason these researches are really irrelevant to our argument. A great deal more research into the teaching of particular processes could be done if (a) it were first made reasonably probable that children were in fact mathematically aware of what was going on, and (b) measuring techniques were devised by means of which the extent of such awareness could be ascertained. Measuring techniques are in fact now being devised by the National Foundation for Educational Research for England and Wales, and a certain number of 'concept tests' are ready in a preliminary form as research tools. (Anyone interested in using these tools should contact the Foundation[1] direct, giving a sketch of the kind of experiment he wants to conduct.)

The work of the Madison Project must be classed under research into method as well as into curriculum, as the introduction of the 'symbol game' is a valuable contribution, and it is important to investigate its role in the general context of mathematics-learning. The Mathematics Individual Learning Experiment at Queens College, New York, uses as its principal variable individual versus class work. They have experimented with what they call 'perceptual devices' in a somewhat haphazard way, but are intending to introduce the use of structured material as another variable in the near future.

The principle of multiple embodiment (perceptual variability) combined with the variation of the greatest possible number of mathematical variables (mathematical variability) is being tested in two major areas in England at the moment. One is Leicestershire, where the Leicestershire Mathematics Project is now involving over half the schools in the county; the other is Surrey, where about seventy or more schools have begun an experiment involving the use of the MAB and the algebra materials. Preliminary results using formal arithmetic, problem arithmetic, concept and anxiety tests in Leicestershire are very encouraging. Full results will be reported in due course in *Educational Research* and elsewhere. A more fully controlled experiment is now being undertaken by the National Foundation in Surrey, where a number of the schools are being used as their own controls. Tests are given to children before any materials are introduced. The same age groups will be tested at regular intervals for three years; in this way the teacher variable will be controlled, and, as no other change will have been introduced in any of the experimental schools, any change in

[1] At 79 Wimpole Street, London, W.1.

the scores over the three years should be attributable to the introduction of the new methods. Multiple embodiment is also being tried out by a group of research workers at the University of Florence, Italy, and in a number of independent and public schools in the States, and a start has been made by a small group in Budapest, Hungary, as well as groups in Adelaide and Melbourne, Australia.

Research into the effectiveness of 'whole' methods brings enormous difficulties. It is not at all certain that a method can be transplanted wholesale from one environment into another rather different one. Transplanting the work from Leicestershire into Surrey was undertaken by a number of people who were active in or at least familiar with the work in Leicestershire, and who gave lectures, demonstrations and so on. After this, two or three pilot schools started work on their own, and this was followed by the larger-scale experiment which began in September, 1961. Observing some of the classes in action, one had to admit that the transplanting had been a success; there was no noticeable fundamental difference between the Leicestershire and the Surrey classes. One naturally notices a great many individual differences moving from school to school. These have always been encouraged and teachers made to feel that they have something important to contribute. The transplanting was probably successful because, apart from the drive and enthusiasm of the initiators, the two environments are similar. An attempt to introduce the methods into an American independent school met with somewhat unexpected and very real difficulties. Difficulties were likewise experienced when the Florence and the Budapest experiments were started. In other words, there are bound to be growing pains in any attempt to introduce a 'whole' method into a different culture, and it is not at all certain that the introduction is always going to be possible or practicable. Apart from this, however, there is the question of expense. A carefully thought out controlled experiment lasting several years is bound to use up considerable funds, and such funds are not readily available for educational projects. The ideal would be to establish an international fund which could allocate its resources to any part of the world where educational projects were planned, preferably in co-ordination with other projects elsewhere. Plans are being made to ensure that at least some of the proceeds from the sale of structured materials are used to cover future expenses of the International Study Group.

The position of research into the cognitive processes underlying the learning of mathematics is really quite simple. The only sizable contribution so far has come from the Geneva school headed by Jean Piaget, where an enormous amount of thought and experiment has gone into developing a theory which is so well known that I need not go into any detail here. There is no theory which does not owe a great deal to the work of the Geneva school. The other important contribution comes from Jerome Bruner, whose work on thinking is also internationally known. It is only recently, however, that Bruner has begun to interest himself in mathematics-learning as a special aspect of the cognitive process. As a result of his developing interest, the Mathematics-Learning Project was formed at the Harvard Cognitive Center. It ran for the length of the academic year 1960–1, and this volume is one of the reports of the year's activities. Bruner has interested himself particularly in mathematical blockages; a report of this work, which ran during the year 1959–60, will be appearing shortly. Inhelder was working at the Center for Cognitive Studies with four- and five-year-olds during the academic year 1961–2, and so one may expect a certain amount of cross-fertilization of ideas between the Center and Geneva as a result of the year's research.

Apart from work in these two major fields, and my own contributions, the work of Skemp at Manchester University has shown a promising lead.[1] His predictions for thirteen-year-olds are more accurate than any previous predictions on percentage passes in mathematics examinations at sixteen. According to him, mathematical understanding depends almost entirely on the exercise of reflective intelligence (approximately what has been called analysis here). One wonders whether if equally mathematical but constructively loaded questions were asked in the examinations the predictions on Skemp's tests would be less accurate. In any case, Skemp's work opens up another avenue of attack in this most intractable field of research.

The work being carried on in Florence by members of the Institute of Psychology of the University of Florence, notably by Ferrara-Mori and Morino, may also turn out to be promising. This, too, is on the borderline between method and underlying process. One study has sought to validate the principle of multiple embodiment by teaching an experi-

[1] Richard Skemp. 'On Sensori-motor and reflective intelligence', *British Journal of Psychology* (April, 1961).

mental group of children in the fourth grade the mathematical concept of the square by multiple embodiment, and teaching this concept to a control group in the conventional manner. There has also been a study of a small group of young children, carefully documented by notes and photographs, to ascertain the stages through which children pass, beginning with undirected free play with structured material such as the MAB, and developing gradually into purposive mathematical activity. An interim account of these experiments is given in Appendix II.

9.7. *The strategy of possible future research*

It is clear from this account of the present state of research (which is by no means exhaustive), that future research should be both interdisciplinary and international. The problems involved in studying the learning of mathematics open up psychological, mathematical and philosophical questions, as well as practical educational questions such as classroom organization, punishment-reward systems and the like. No one person can be equally conversant in all these disciplines. It is also clear that there is so little fundamental work in progress relevant to the evolution of satisfactory models that co-ordination of effort is an urgent necessity. This must perforce be done on an international level which, as I have emphasized, means spending money. As money is not readily available for international schemes (capital in each country is usually interested in supporting work that immediately and directly benefits that country), a skeleton organization with minimum funds would have to be mounted to co-ordinate work and create a working international unit. This unit could possibly publish its own journal, thus acquiring reputation for its work, and eventually attracting capital, either from private or public sources. In this way the framework for international and interdisciplinary research may be created. It is encouraging that those people who are intimately concerned with the work, whether from the point of view of curriculum, methods of teaching, or the underlying psychology, are finding it possible to meet each other and work together on problems. Both the Geneva school and the Harvard Center for Cognitive Studies are acting as international clearing houses for information, as well as carrying out fundamental research. The point to bear in mind is that research alone will not meet the demands of the situation, however essential such research might be. The psychologists will have to listen not only to the

mathematicians, as they have begun to do already both in Geneva and at the Harvard Center, but also to practising teachers, training-college lecturers, administrators of school systems and so on who may need to start their own centres where work relevant to their special roles may be satisfactorily studied. The School Mathematics Study Group is one such centre, and is already to some extent interdisciplinary in structure. It will almost certainly play an international part in some form or another in the future, if only by making its findings available to similar centres of research elsewhere.

One difficulty to be faced is the difference in educational climate in the various countries. Something which may be the order of the day in one country would quite probably be regarded as old-fashioned and outworn in another, and revolutionary and impracticable in yet another. Often people have to see things to believe that they are possible. Research workers in the field who have been investigating the possibility of teaching young children advanced mathematics have found it impossible to convince others of their results without taking them into the classroom or showing them a film or a tape-recording. Attitudes tend to get set; the known way is regarded as the natural or indeed the only way.

On the other hand, sometimes possibilities do not occur to research workers, simply because they are out of touch with classroom conditions. However, the gradual seeping down of new discoveries is a natural historical process, and one that will have to be accelerated as knowledge advances more rapidly. The calculus is only 300 years old, yet it is now taught to fourteen- and fifteen-year-olds in England (albeit not very effectively). Group theory was only admitted to the syllabus of London University during the last war, and we now find that it is possible to teach this to twelve-year-olds. The whole theory is barely a hundred years old. Examples of this seeping process, through university to the higher and then lower grades of the schools, are getting more frequent. But here again, the process has been allowed to proceed quite at random, without any attempt on anybody's part to put it on a systematic basis, either scientifically or historically. If people could be made more aware of this historical process in curriculum-change, the changes of attitude I mentioned earlier would probably be easier to bring about; this in turn would help us to control the process to everybody's advantage.

The differences in the educational climates of different countries do not show only in resistance to curriculum-change. The extent to which teaching is formal varies considerably from country to country, not to mention within the same country. And the degree to which an educational system is centralized or regionalized must also be kept in mind. Experimenting is difficult in a centralized country, as the Minister of Education usually has to grant permission for any radical departure from established practices, and so decisions are bound to be influenced by political considerations—not always the happiest influences on educational developments! Also, once a decision is made, large numbers are immediately involved; the price of failure is very great, and so is the premium on hiding failure by means of all sorts of humbug. An inhabitant of a regionalized country may laugh at these difficulties; they are nevertheless very formidable in a centralized one.

So traditions in curriculum and teaching method and the organization of the educational system must be taken into account whenever a change is to be initiated. Everyone will agree that persuasion is always better than compulsion as a method of bringing about change. The best way to persuade people that something is possible and desirable is to show them how it can be done. The pilot groups working now in America, Italy and Hungary on the idea of using multiple embodiments and full mathematical variation of the mathematical variables are, it is hoped, going to develop into prototype groups acting as models on which other educational exports can be based. These groups will have to digest the new ideas and assimilate them into the local climate by retaining the essentially good and characteristic part of the developing culture-pattern, and yet showing that the soil is fertile enough for the new seeds to grow and to adapt themselves to new environment. When the local pilot groups have done their work of assimilation it must be possible for people to visit the experimental areas; failing this, films and tape-recordings can act as a substitute. The Madison Project has used tape-recordings from the very beginning, not only to show what can be achieved by a single group, but also as documented evidence on how the results were obtained. The Massachusetts Institute of Technology Film Studio have done a great deal to propagate new ideas on science teaching by making films which show practising teachers what can be done and how. With modern methods of communication, spreading information should not be difficult once the rather more difficult task

of adaptation has been completed. Given the will to co-operate on an international and interdisciplinary scale, it is quite certain that the physical means of solving the pressing problems which surround mathematics-learning will become available.

Further Reading

I do not feel it would be helpful to give a long bibliography of this vast subject, and in any case no such list of relevant books and publications could ever hope to be complete. The following suggestions are, however, made for further reading.

Bartlett, Sir Frederick. *Thinking*, Allen & Unwin (London, 1958). Starting with the assumption that thinking develops in the same way as other high-level skills, the book analyses the process of thinking into the closed (scientific), empirical, everyday and adventurous (artistic?) kinds.

Bruner, J. S. *The Process of Education*, Harvard University Press (1960). An evaluation of the educative process in the light of present knowledge of the thinking process. A great challenge to all educators.

Bruner, J. S., Goodnow, J. J. and Austin, G. A. *A Study of Thinking*, John Wiley & Sons (New York, 1956). A detailed study of the strategies observed in artificially produced conjunctive, disjunctive and other logical classification of attributes. Includes a fascinating appendix by Roger Brown applying the technique to the explanation of the growth of language in a child.

Chomsky, N. *Syntactic Structures*, Mouton (The Hague, 1957). Pioneering work on generative grammars and syntax. How do we form our syntactical classes and the rules of construction that 'permit' us to say certain things and not others? These and other questions fully discussed.

Dienes, Z. P. *Building Up Mathematics*, Hutchinson (London, 1960). An introduction to the constructive approach to the teaching of mathematics, mostly at the primary level, but with mathematics usually classed as secondary school curriculum. The use of structured materials of various kinds is explained in detail.

Dienes, Z. P. *Concept Formation and Personality* (Leicester University Press, 1959). A study of the dynamics of abstract concept formation, in the light of open and closed thinking, construction and analysis, implicit and explicit thinking, etc., and an attempted tie-up with

personality variables using a projective technique. Mathematical groups with ten-year-olds.

Dienes, Z. P. *The Power of Mathematics*, Hutchinson (London, 1963). An attempt to bridge the gap between the essentially constructive thinking of the pre-adolescent child and the analytical modes of thinking at the onset of the hypothetical-deductive stage.

Dienes, Z. P. 'On abstraction and generalization', *Harvard Educational Review* (Summer, 1961). A theoretical discussion of the functions of abstraction and generalization as seen in the learning of mathematics. 'Learning cells' organically joined in a 'live' thread of thinking are proposed as a suitable model in terms of which the observed facts may be explained.

Gréco, R. and Morf, A. *Structures Numériques Élémentaires*, Presses Universitaires de France (1962). Reports recent experiments by the authors on 'conservation'.

Hale, T. and McRoy, E. 'UICSM's decade of experimentation', *The Mathematics Teacher*, Vol. LIV (December 1961). Short account of project.

Hayeck, F. A. *The Sensory Order* (Routledge & Kegan Paul, London, 1952). A new and daring theory proposed on the origin and nature of sensation.

Hendrix, Gertrude. 'Learning by discovery', *The Mathematics Teacher*, Vol. LIV (May, 1961). On non-verbal insights in mathematics.

Hendrix, Gertrude. 'The psychological appeal of deductive proof', *The Mathematics Teacher*, Vol. LIV (November, 1961). An appeal for the encouragement of the 'intuitive leap' as the essential prerequisite to deductive rigour. It is contended that group, ring and field properties are much more suitable for *really* rigorous treatment by schoolchildren than the pseudo-rigour employed in the usual treatment of deductive geometry.

McKellar, Peter. *Imagination and Thinking*, Cohen & West (London, 1957). An experimental study of the function of images in thinking. A must for future 'image-manipulators'.

Page, David A. *Number Lines, Functions and Fundamental Topics* (circulated by the University of Illinois Arithmetic Project, 1961). A book crammed full of essential information for every mathematics teacher on the possibility of playful manipulations of mathematical structures. The 'symbol game' at its best.

Piaget, Jean. *Le Genèse du Nombre chez L'enfant*, Neuchatel (Paris) or *The Child's Conception of Number* (London, 1952). This is now a classic, in which the main positions of the Geneva group are explained.

Piaget J., Grize, Gréco et al. *Sur la Construction du Nombre*, Neuchatel (Paris, 1961). This includes an exposition by Piaget of the present position taken up by the Geneva group, the formalization of the theory of *groupements* and a detailed account of Gréco's experiments on the stages through which the synthesizing of the cardinal and ordinal aspects of recurrence pass.

Read, Sir Herbert. *Education through Art*, Faber & Faber (London, 1958). A comprehensive exposition of the process of education as integration of the personality through the assimilation of artistic experience.

Rózsa, Peter. *Playing with Infinity*, G. Bell & Sons (London, 1961). The author introduces the layman into the delights and difficulties of mathematical thinking from simple addition and its properties to the calculus and formal logic.

Sealey, L. G. W. and Gibbon, V. *Communications and Learning*, Basil Blackwell (Oxford, 1963). An impassioned appeal for humanity and understanding in our over-formalized classes.

Skemp, R. R. 'Reflective intelligence and mathematics', *British Journal of Educational Psychology*, Vol. XXXI, pp. 45–55 (1961). A novel method of describing and assessing the mathematical activities of schoolchildren.

Stern, Catherine. *Children discover Arithmetic*, Harper (New York, 1949). A stimulating book in which the use of structured materials devised by the author is explained; based on Gestalist assumptions.

APPENDIX I

Description of the experimental groups

Group A Construction Group

The children were taken from a Cambridge (Mass.) independent school, third grade. There were two boys and two girls, all eight years old. A room was specially allocated for the sessions and each child had a table in a corner. Each had an observer; Zed Dienes was present at all the sessions and Jerome Bruner at most of them. The group met four times a week for about six weeks.

The mathematical material presented on the instruction cards included the following: commutative law of multiplication; associative law of multiplication; spatial study of triple products; distributive law of multiplication leading to the formulae $(X+1)^2=X^2+2X+1$ and $(X+2)^2=X^2+4X+4$; and some attempts at cubics.

The physical material consisted of squares and triangles, large and small; rectangles to fit along the sides of the large squares; pegs and pegboards; cups, and beans to put in them; and balance beams with hooks at equal distances on either side of the fulcrum, with metal rings to place on the hooks. The tables had green chalkboard finish on them, and the children were encouraged to draw either on their own desks or to go to the blackboard at any time to work things out.

The instruction cards were framed to induce the children to carry out tasks that would result in their eventually constructing their own mathematical concepts. Little analytical work was required by the instruction cards; if there was a certain amount, it was meant to be done by manipulating the physical material.

Group B Arithmetic Group

The children were taken from a second grade of a Cambridge independent school. Sometimes the whole class was taken, although children worked individually most of the time; at other times the class was divided into groups of seven children.

The mathematical concept to be learned was the one of place value in a number of bases including the base ten.

The physical material used was the MAB of both the rectangular and the triangular-trapezoidal forms. Trees with the same number of branches going off at each branch point were also used as a check on the abstraction. Bases 2, 3, 4, 5 and 10 were used, going up to the sixth power of 2, the third power of 10 and the third powers of the other bases.

There were no observers available for this group and notes were taken after each session. Not more than three adults were ever present at the sessions, which were held twice a week for about six weeks.

Group C Logs and Indices Group

Two boys and two girls were taken from the fourth grade of a Cambridge independent school. There was one 'teacher' and one observer at each session. The sessions went on for about eight weeks, meeting twice or sometimes three times a week.

The mathematical material was integral and fractional positive exponents (powers and roots and logarithms); applications to geometrical progressions and to similar figures in connexion with the area and the volume theorems.

The physical material consisted of wooden blocks whose volumes were in geometrical progression to the bases 2, 3, 4, 5, 8, 9, 27, 32, 64. There was one rectangular set, and one triangular-trapezoidal set.

Instructions were given mainly orally.

Group D First Games Group

Six fourth-grade children, three boys and three girls, were taken from a Newton public school for a period of four weeks in a room by themselves, with three observers and a 'teacher'.

The mathematical material included mathematical groups with up to eight elements; sub-groups; multiplication matrix of a group; vector spaces up to four dimensions; and the construction of the complex algebra and another two-dimensional algebra.

The physical material included coloured triangles and hexagons (for the group games), squares, books, pieces of paper with arrows, boxes, etc., in fact anything that came to hand to represent dimensions in a vector space.

Group E Second Games Group

Seven boys and seven girls were taken from a sixth grade of a Cambridge independent school. There was one 'teacher' and one observer.

The mathematical material was the same as in Group D; transformations of vector spaces were added, leading to matrices and their inverses (up to 3 by 3 matrices); also four-dimensional algebras were constructed apart from two- and three-dimensional ones. Characteristic values were also found by some children.

The physical material was the same as in Group D.

PARTICULARS OF THE CONCRETE MATERIALS USED IN THE EXPERIMENTAL GROUPS

Algebra Experience Materials (or AEM)

The unit of length in these materials was $\frac{3}{4}$ of an inch. The 'two-dimensional' material was all $\frac{3}{8}$ of an inch thick. The following two-dimensional material was used:

A quantity of 1×1 squares; unit equilateral triangles; and rhombi with sides of unit lengths, angles 60° and 120°. Strips $5\frac{1}{2} \times 1$, also $7\frac{1}{2} \times 1$. Squares $5\frac{1}{2} \times 5\frac{1}{2}$, $7\frac{1}{2} \times 7\frac{1}{2}$.

There were also unit cubes (i.e. $\frac{3}{4}$ by $\frac{3}{4}$ by $\frac{3}{4}$ inch cubes); pegboards with holes in a square lattice, as well as with holes in a 60° lattice, forming adjacent equilateral triangles, and pegs of various colours to fit in the holes; balance beams with hooks at equal distances from each other, 9 or 10 on each side of the fulcrum, metal washers to fit on the hooks, and elastic to enable a large number of washers to be hung on the same hook; a certain number of equilateral triangles with sides 2, 3, 4, 5, 6 and 7 units long, all $\frac{1}{2}$ unit thick; matchboxes of various sizes; paper cups, and beans to put in them.

Logs and Indices Blocks

The unit length here was $\frac{3}{8}$ of an inch. There were pieces of wood of the following dimensions:

Base 2. $1 \times 1 \times 1$, $1 \times 1 \times 2$, $1 \times 2 \times 2$, $2 \times 2 \times 2$, $2 \times 2 \times 4$, $2 \times 4 \times 4$, $4 \times 4 \times 4$, $4 \times 4 \times 8$, $4 \times 8 \times 8$, $8 \times 8 \times 8$ (2 of each piece).

Base 3. $1\times1\times1$, $1\times1\times3$, $1\times3\times3$, $3\times3\times3$, $3\times3\times9$, $3\times9\times9$, $9\times9\times9$, $9\times9\times27$, $9\times27\times27$, $27\times27\times27$ (3 of each).

Base 4. $1\times1\times1$, $1\times1\times4$, $1\times4\times4$, $4\times4\times4$, $4\times4\times16$, $4\times16\times16$, $16\times16\times16$ (4 of each).

Base 5. $1\times1\times1$, $1\times1\times5$, $1\times5\times5$, $5\times5\times5$, $5\times5\times25$, $5\times25\times25$, $25\times25\times25$ (5 of each).

Base 2 was often completed with some $8\times8\times16$, $8\times16\times16$ and $16\times16\times16$ pieces to bring this set up to the twelfth power.

Multibase Arithmetic Blocks (MAB)

The unit length here was also $\frac{3}{8}$ of an inch. There were wooden blocks whose dimensions were

$1\times1\times1$, $1\times1\times N$, $1\times N\times N$, $N\times N\times N$

and a considerable number of each kind, N being equal to 3, 4, 5, 6 and 10 depending on the base being worked.

These blocks were all marked on all surfaces with 1×1 squares notched in the wooden surface. Every piece was a rectangular parallelepiped (some of course being cubes).

A non-rectangular version of this material is shown on the accompanying diagrams, where the bases 3, 4 and 6 were embodied.

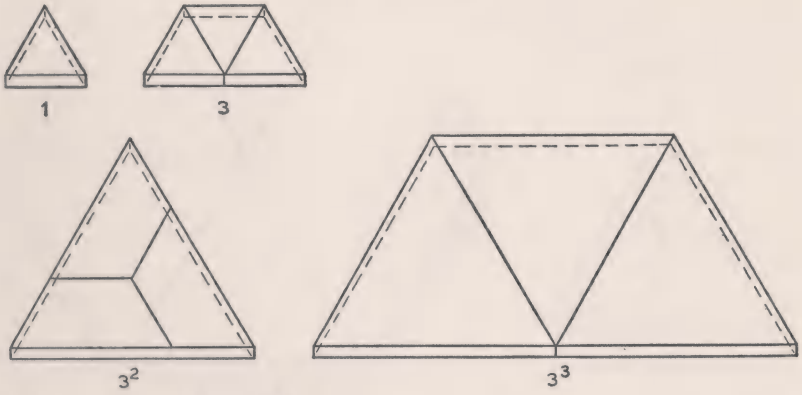
Multibase Tree Games

These consisted of 'trees', starting on the ground with N branches which at a certain level each branched off into N further branches, and so on. As the last lot of branches were to represent the units, i.e. the zero-th power, the lowest branches represented the highest power. Trees were used with $N=2$, 3, 4, 5 and 10. With $N=2$, the highest power was 6, with $N=3$, 4 and 5, the highest power was 3, and with $N=10$, the highest power was 2.

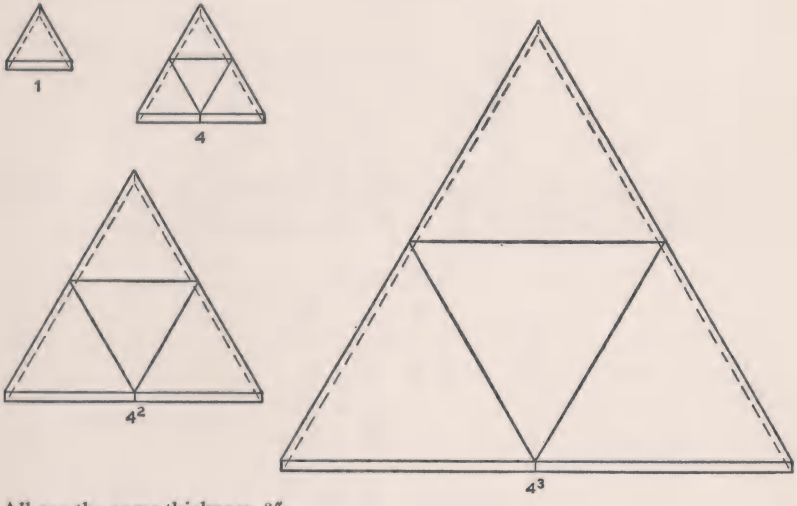
Groups and Vector Spaces Material

Any objects that came to hand were used as long as there were quite a number of them available, and a suitable opposite or opposite position could be found to each kind. Boxes, matches, cups, coloured counters, coloured triangles and hexagons, books, pieces of cardboard with coloured arrows on them, etc., were all used. Bodily displacements by the children themselves were also used in establishing the relationships between elements of groups.

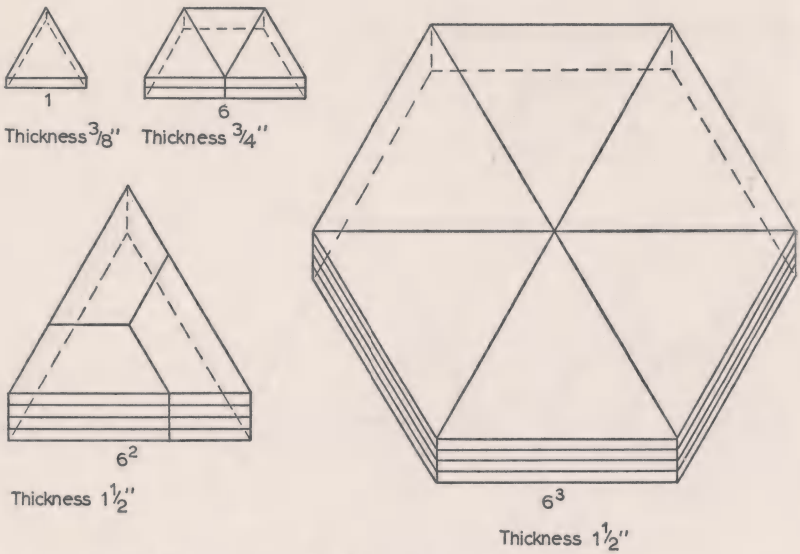
Triangular-trapezoidal Arithmetic Blocks



All are the same thickness, $\frac{3}{8}$″



All are the same thickness, $\frac{3}{8}$″



A summary of the games played by Groups D and E

(1) *2-move game.* The moves here are 'change' and 'no change', denoted by C and N. The matrix for the game is

Second move \ First move	N	C
N	N	C
C	C	N

(2) *3-move game.* The moves can be typified by a 'no move', a 'left' and a 'right', denoted by N, L and R. The matrix of the game is:

Second move \ First move	N	L	R
N	N	L	R
L	L	R	N
R	R	N	L

(3) *First 4-move game.* Apart from 'no move', there are three 'real moves'. The same move done twice equals the 'no move,' any two 'real moves' equal the third 'real move', and the 'no move' has no effect. The matrix of the game is:

First move

Second move	N	A	B	C
N	N	A	B	C
A	A	N	C	B
B	B	C	N	A
C	C	B	A	N

(4) *Second 4-move game.* Apart from the 'no move', the moves in this game can be typified by a quarter-turn left, a quarter-turn right and a half-turn. Denoting these by L, R and H respectively, we have the matrix:

First move

Second move	N	L	H	R
N	N	L	H	R
L	L	H	R	N
H	H	R	N	L
R	R	N	L	H

(modulo 4 addition)

(5) *First 6-move game.* Apart from the 'no move', the moves in this game can be typified by a 60° turn to the right, a 60° turn to the left, a 120° turn to the right, a 120° turn to the left, and a half-turn. Denoting these by N, R, L, 2R, 2L and H, we have the matrix overpage:

First move

Second move	N	L	2L	H	2R	R
N	N	L	2L	H	2R	R
L	L	2L	H	2R	R	N
2L	2L	H	2R	R	N	L
H	H	2R	R	N	L	2L
2R	2R	R	N	L	2L	H
R	R	N	L	2L	H	2R

(modulo 6 addition)

(6) *Second 6-move game.* This arises most easily out of the permutation of three objects. They may be imagined at the three vertices A, B, C of a triangle. Moving round one vertex by all objects to the right will be R, the corresponding left move will be L, the exchanging of pairs will be AB, AC and BC, the 'no move' being again N. The matrix is as follows:

First move

Second move	N	L	R	AB	AC	BC
N	N	L	R	AB	AC	BC
L	L	R	N	BC	AB	AC
R	R	N	L	AC	BC	AB
AB	AB	AC	BC	N	L	R
AC	AC	BC	AB	R	N	L
BC	BC	AB	AC	L	R	N

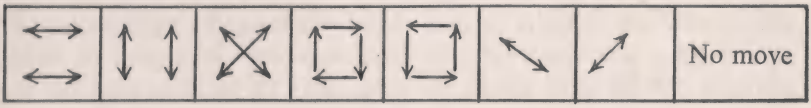
This is the first non-commutative game. For example, (AB)(AC) = R and (AC)(AB) = L. There is a commutative sub-game consisting of N, L, R, yielding the 3-move game.

(7) *First 8-move game.* This game can be typified by imagining any three objects in one of two positions. The moves between these positions will be the moves in the game. For example persons A, B and C could be standing up facing either North or South. The moves consist in none, or some or all the people doing half-turns. This gives the 8 moves N, A, B, C, AB, AC, BC, ABC. The matrix is as follows:

First move

Second move	N	A	B	C	AB	BC	ABC	AC
N	N	A	B	C	AB	BC	ABC	AC
A	A	N	AB	AC	B	ABC	BC	C
B	B	AB	N	BC	A	C	AC	ABC
C	C	AC	BC	N	ABC	B	AB	A
AB	AB	B	A	ABC	N	AC	C	BC
BC	BC	ABC	C	B	AC	N	A	AB
ABC	ABC	BC	AC	AB	C	A	N	B
AC	AC	C	ABC	A	BC	AB	B	N

(8) *Second 8-move game.* This game can be embodied by taking a rectangular piece of paper and putting a red arrow on one side of it and a blue one, say, on the other, each arrow being parallel to the sides of the paper. The moves are 'no move', 'quarter-turn right', 'quarter-turn left', 'half-turn', 'flip over away from you', 'turn over sideways', 'flip over diagonally to right', 'flip over diagonally to left'. It can also be embodied by four people standing at four corners of a square and doing the following moves:



The matrix of this game is as follows:

First move

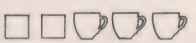
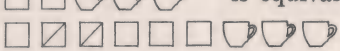
Second move	e	a	b	c	ϵ	α	β	γ
e	e	a	b	c	ϵ	α	β	γ
a	a	b	c	e	α	β	γ	ϵ
b	b	c	e	a	β	γ	ϵ	α
c	c	e	a	b	γ	ϵ	α	β
ϵ	ϵ	γ	β	α	e	c	b	a
α	α	ϵ	γ	β	a	e	c	b
β	β	α	ϵ	γ	b	a	e	c
γ	γ	β	α	ϵ	c	b	a	e

There are two 4-move sub-games in this 8-game move. The first 4-move game is given by the moves (e, a, b, c), the second 4-move game by (e, b, ϵ, β). Both of these are commutative sub-games, although the entire game is non-commutative.

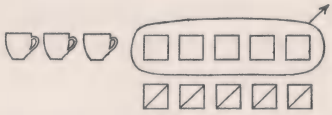
Vector-Space Game

Any number of objects of any number of varieties may be chosen for this game. The number of varieties will be the number of 'dimensions' in the space. For example a three-dimensional space would be defined by, say, cups, boxes and triangles. A vector is a collection of allowable objects. The identity principle is defined for vectors by considering 'opposite' objects as cancelling each other out; this necessitates the definition of an opposite to each object. Cups can be right-way-up or upside-down, these can be opposites, boxes can be closed or open, triangles can be with vertex pointing away or pointing towards us,

which can be opposites. Opposite pairs are regarded as equivalent to zero, and so for example

the vector  is equivalent to
the vector 

Addition of vectors is defined by joining other vectors, subtraction by removal of vectors. It is always possible to subtract, as opposite pairs can be introduced. For example

3 cups—5 open boxes: 

'Cover-story' embodiments of vector spaces can be easily found. Ong is the dance-story, for example; the other the mouse-hamster-guinea-pie farm.

The dance-story is as follows: some children and some adults go to a dance. There is a rule that adults can only dance with adults and children with children. There is another rule that you must dance if there is anyone you could be dancing with. For example, 3 girls and 5 boys would not be allowed to sit out, because 3 of the 5 boys would have to ask the 3 girls to dance. Those that cannot find partners go to the refreshment room and have drinks. The number of children and adults in the refreshment room at any given time form the vectors, and the opposites are the opposite sexes. Theoretically there should be an infinite number of couples dancing all the time, to make every subtraction always possible.

Algebra Games

In the vector-space game only additions and subtractions are possible. The game can be turned into an algebra game by defining multiplication of vectors by other vectors. The procedure is to stipulate the distributive law and to define multiplication of 'unit vectors' by each other. A unit vector is a vector in which there is only one object. In a three-dimensional vector space there are of course 6 unit vectors, as each object has two positions. This necessitates the definition of a 6 by 6 multiplication table. The games embodying mathematical

groups are an obvious resort for solving the problem of how the unit vectors should be multiplied, although of course the choice is theoretically entirely arbitrary. Algebras obtained by filling in tables by random choices of values for the products tend to be extremely complicated.

A two-dimensional vector space will need a 4-move game to turn it into an algebra. Therefore there are two algebras we can have ready-made for the two 4-move games. One of these is the complex algebra.

'Cover-stories' can be made up for these algebras too. The dance-story can be thought of as taking place in some magic land where all the people on the dance-floor have magic powers over those in the refreshment room. Magic words uttered from the dance-floor could be said to have the following effects:

Boy's magic = Nothing happens.
Girl's magic = Everyone in the refreshment room changes sex but keeps the same age.
Man's magic = Children become adults of the same sex, adults become children of the opposite sex.
Woman's magic = Children become adults of the opposite sex, adults become children of the same sex.

Simultaneous magic performed by several dancers at the same time will make all the magic happen at once, and so the final number of people in the refreshment room will be that multiple of the original number which is the number of magicians using magic. In this way the boys can do magic, for if two boys do their magic simultaneously, the number of people is simply doubled. This is necessary if we want to bring in binomial multipliers. The magic in the dance corresponds to the screen in the mouse-hamster-guinea-pig farm game.

Complex Algebra Transformation Game

One child sits at each of the four small tables, and the remaining two at the long table. Suppose Jon and Ross are at the long table, they are going to choose the heaps (vectors). Then let Larry, Sandy, Paula and Collette stand at their respective tables. Each one of these four will have different jobs, and each job will depend on what Jon and Ross have put on the long table. Let us have closed and open boxes, and right-way-up and upside-down cups again. Opposites cancel out as usual. Let the 'jobs' be as follows:

Larry's job He puts on his table exactly what is on the long table (identity transformation).

Sandy's job She puts on her table the opposite of every object that is on the long table (reversal transformation).

Paula's job She replaces what is on the long table as follows:
For every closed box on the long table, she puts a right-way-up cup on her table.
For every open box on the long table, she puts an upside-down cup on her table.
For every right-way-up cup on the long table, she puts an open box on her table.
For every upside-down cup she puts a closed box on her table.

Collette's job She replaces as follows:
For every closed box she puts an upside-down cup on her table.
For every open box she puts a right-way-up cup on her table.
For every right-way-up cup she puts a closed box on her table.
For every upside-down cup she puts an open box on the table.

It will be seen that Paula and Collette were given the 'i' and '$-$i' multiplications of the complex algebra.

APPENDIX II

Summary of research on mathematics-learning in an Italian university

The Institute of Psychology of the University of Florence, under the direction of Professor Alberto Marzi, has carried on since 1958 a programme of research on the formation of mathematical concepts.

The research aimed both at showing the existence of preliminary processes leading to the acquisition of mathematical concepts (taking Piaget's hypotheses as a starting point), and at analysing the mental processes leading to the mathematical structure of thinking.

As far as the genetical aspect of the formation of mathematical concepts is concerned the research was carried out on six-year-old children —that is, pre-school children—who were followed during the whole school year and are still under observation. In its other aspects, the research was carried on through the system of problem-solving, using both introspection and behaviour analysis.

The brief reviews that follow are a summary of the activity of the Institute of Psychology during the period 1958–61.

1. *The constructive bases of logico-mathematical processes in thinking* (Francesca Morino-Abbele)

The aim of this research was to verify and stress the importance of the constructive bases that precede the acquisition of and pave the way to the learning process of mathematical concepts.

The materials used were the MAB, devised and prepared by Zed Dienes following a specific theory on mathematical-learning.

The research was carried out on ten six-year-old children—pupils in a first-grade class—who had received, as far as possible, no traditional information about mathematics. It was carried on during the whole academic year 1960–1. All the children underwent intelligence and personality tests.

The method used in the research was based on the observation of their behaviour, by taking notes and photos of what the children were doing. The materials were given out with no particular instruction; they were simply told to do what they wanted with them.

Two specific problems were taken into consideration:

(a) Which and how many stages are necessary to prepare the learning process?

(b) How does one reach this learning process?

The stages observed so far are five.

Through a 'free play' activity the child reaches a phase of 'structured activity' in which he strives for geometrical shapes. He then reaches a stage of 'perception of quantity' or 'quantitative stage', which is his first approach to the concept of 'number'.

The children organize their 'work' following patterns that range from an orderly succession of all stages, with very detailed construction work, to the adoption of intuitive solutions, which are often related to a disorderly and unstable type of activity.

On the other hand, some children showed a marked preference for handling the materials for the sake of handling them, and reached the solutions through successive, completely random, efforts.

A communication on the present research was among the contributions to the National Congress of Psychology held in Palermo, Italy, in April 1961.

2. *Perception of geometrical gestalt and learning of mathematical concepts* (Gina Ferrara-Mori and Francesca Morino-Abbele)

It has been the aim of this research to study the learning process of mathematical concepts and verify whether adequate perceptive aids might make this learning process easier.

The pupils of four classes, two all-boys and two all-girls respectively, all in the fourth elementary grade, were selected for this research. Two classes were taken at random as the experimental group while the remaining ones were used as the control group.

The learning stage was based on exercises containing basic mathematical principles; geometrical materials as varied as possible were supplied to the experimental classes, while number and traditional systems were used in the control classes. The exercises were prepared

each time after observation of the results obtained in the previous stage. Over this period—about two months—the experimenters took the place of the teachers during the experiments.

At the end of this period, the children took a test—the same for all children—prepared to contain the basic principles of mathematics they had been taught during the preparatory stage.

The analysis of the results has shown clearly that the experimental classes, those pupils who had worked in constructive situations with handling of geometric forms, had reached a better understanding of the mathematical concepts and of most of their possible logical implications and extensions.

Though we are far from a thorough knowledge of the mental structures leading to mathematical insight, it is obviously necessary to set about the work this way.

A communication on the present research was among the contributions to the National Congress of Psychology held in Palermo, Italy, in April 1961.

3. *Analytical and intuitive procedure in a problem-solving situation* (Gina Ferrara-Mori)

The starting point of this research is represented by the theories of Bruner, Bartlett and Dienes on individual differences in the formation of concepts, i.e. on the different ways of organizing thought processes when trying to find some logical relations in a problem-solving situation.

This research tries to determine two different processes that might be adopted in the solution of a problem (Bartlett): a constructive or intuitive one, which leaves room for 'jumps' or 'skips' in the logical relations leading to it, and an analytical one, which makes use of disjunctions, conjunctions, associations and successive efforts as the necessary 'steps' towards the same and only solution of the problem.

Fifty subjects, students entering university, have been examined with the help of a typical Bartlett problem (the so-called 'hidden problem'), which can be solved through very elementary processes of arithmetic, but was in fact solved by only thirty-one students.

The analysis of the results provided many interesting points for discussion, of which the most important are:

How is previous knowledge used?

The importance of conventional procedures;
The detection of a mistake does not automatically imply that it will be corrected;
The importance of the concepts of 'key-move';
How are intuitive 'jumps' made possible?
Where does the analytical process lead?

These points, which are related to the specific problem, but only partially bound to it, may lead to some hypotheses about traits of the child's personality which are directly involved in the process—among these a 'lack of flexibility' and the resulting feeling of 'frustration' which may even cause a complete block. Moreover, an example of a possible measurement of the intuitive 'jumps' can be offered through a mathematical diagram, which studies the steps developed from the initial positions of Bartlett's problem.

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